Remarks on a model of A.Majda for combustion waves

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Rapports de Recherche

N° 281

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Mars 1984

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ABSTRACT In this paper, we study the system :

$$\begin{cases} u' = su - sv + f(u) + C_0 , \\ dv'' + sv' = K\,\Phi(u)v . \end{cases}$$

These equations modelize combustion waves, with a positive diffusion coefficient d. The system with d = 0 was introduced and investigated by A. MAJDA [7] . We extend the results of [7] (existence, uniqueness and some qualitative properties) to the case d > 0, using different arguments. We show the continuous dependence of the solutions with respect to d, and study the limit $d \to 0$.

REMARQUES SUR UN MODELE DE A. MAJDA POUR DES ONDES DE COMBUSTION

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RESUME Nous étudions dans ce rapport le système :

$$\begin{cases} u' = su - sv + f(u) + C_0 , \\ dv'' + sv' = K\,\Phi(u)v . \end{cases}$$

Ces équations sont liées à des modèles d'ondes de combustion, avec un coefficient de diffusion positif d. Ce système avec d = 0 a été introduit et étudié par A. MAJDA [7] . Nous étendons les résultats de [7] (existence, unicité et quelques propriétés qualitatives) au cas d > 0, par des méthodes différentes. De plus, nous montrons la dépendance continue des solutions par rapport à d, et étudions le passage à la limite $d \to 0$.

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SECTION I : INTRODUCTION - MAIN RESULTS

In recent work, A. Majda proposed a simplified model for the qualitative study of one-dimensional combustion waves : see [7]. This model is the following system of equations :

$$(1.1) \qquad \begin{cases} (u + q_0 Z)_t + (f(u))_x = \beta u_{xx} \\ Z_t = - K_0 \, \Phi(u) Z \end{cases}$$

In these equations, u is a lumped variable having some features of density, temperature and velocity, and Z is the mass fraction of unburnt gas. q_0 is the energy liberated by the chemical reaction ($q_0 > 0$). The reader is referred to [7] for the signification of the other variables. Φ and f satisfy the following assumptions (an ignition temperature mechanism is assumed) :

$$(1.2) \qquad \begin{cases} \Phi \in C^1(\mathbb{R}) \; ; \; 0 \leq \Phi \leq 1 \; ; \; \Phi' \geq 0 \; ; \\ \Phi(0) = 0 \; ; \; \forall\, u > 0, \;\; \Phi(u) > 0 \; ; \\ \exists\, U_0 > 0, \;\; \Phi(U_0) = 1 \; ; \; \forall\, u < U_0 \, , \;\; \Phi(u) < 1 \, . \end{cases}$$

$$(1.3) \qquad \begin{cases} f \in C^2(\mathbb{R}) \\ \forall\, u \in \mathbb{R} \, , \; f'(u) = a(u) > 0 \, , \; f''(u) > 0 \, . \end{cases}$$

The model (1.1) is derived from the one dimensional combustion equations written in Lagrangian coordinates for a simple reactant → product mechanism (see [7]). R. Rosales and A. Majda [9] and P. Fife [6] have actually shown that these Lagrangian equations with certain additional hypothesis can formally be reduced to A. Majda's model (1.1). In fact, it can be hoped that this qualitative model retains most of the essential features of the Lagrangian equations, except the species diffusion (see [7]). In the present paper, we study the following model, with a positive diffusion coefficient D :

$$(1.4)\quad \begin{cases} (u + q_0 Z)_t + (f(u))_x = \beta u_{xx}, \\ \qquad\qquad Z_t = - K_0\, \phi(u) Z + D Z_{xx}. \end{cases}$$

The aim of this paper is to study the travelling wave solutions $u(x - st)$, $Z(x - st)$ of equations (1.4) . For model (1.1), this work has been completely done by A. Majda [7]. We will also study the relationship between the two models by deriving an asymptotic analysis for small values of the diffusion coefficient D.

For system (1.4) travelling waves propagating with a positive speed s are given by :

$$(1.5)\quad \begin{cases} \beta u' = - s(u + q_0 Z) + f(u) + C_0, \\ D Z'' + s Z' = K_0\, \phi(u) Z, \end{cases}$$

where the prime ' denotes differentiation with respect to $\zeta = x - st$; C_0 is a real constant. Using (1.2), the natural conditions associated with equations (1.5) are :

$$Z(-\infty) = 0 \ ; \ Z(+\infty) = 1 \ ; \ -\infty < u(+\infty) \le 0 < u(-\infty) < +\infty.$$

Following A. Majda [7], we will mainly deal with the more interesting case where s and C_0 are chosen so that the equation

$$(1.6)\qquad - su + f(u) + C_0 = 0$$

has two positive solutions. Therefore, we consider two fixed points A, B with the hypothesis :

$$(1.7)\qquad 0 < U_0 < A < B,$$

and we define :

$$s = \frac{f(B) - f(A)}{B - A} > 0,$$

$$C_0 = sB - f(B) = sA - f(A),$$

$$(1.8)\qquad \hat{q}_0 = \frac{f(0) + C_0}{s} .$$

Under hypothesis (1.3) on f, this implies (see [7]) :

$$(1.9)\qquad 0 < a(A) < s < a(B),$$

$$(1.10)\qquad \hat{q}_0 > 0$$

$$\forall\, q_0 \geq \hat{q}_0,\ \exists!\, \mathcal{U}(q_0) \leq 0,\ -s\,\mathcal{U}(q_0) + f(\mathcal{U}(q_0)) - sq_0 + C_0 = 0$$

Finally we denote : $q_0 Z = v$; $K_0\beta = K > 0$; $D\beta^{-1} = d \geq 0$; $\zeta\beta^{-1} = y$. The problem derived from (1.4) to be addressed here is the following :

$$(1.11)\qquad \boxed{\left\{\begin{array}{l} \text{For } q_0 \geq \hat{q}_0, \text{ find } (u,v) \in \mathcal{C}^2(\mathbb{R}) \times \mathcal{C}^2(\mathbb{R}), \text{ such that :} \\ \qquad u' = -su - sv + f(u) + C_0, \\ dv'' + sv' = K\phi(u)v, \\ \qquad v(-\infty) = 0 \quad , \quad v(+\infty) = q_0, \\ \qquad u(-\infty) \in \{A,B\} \quad , \quad u(+\infty) = \mathcal{U}(q_0). \end{array}\right.}$$

A solution (u,v) of (1.11) satisfying $u(-\infty) = A$ will be referred to as a weak detonation profile. In contrast, if $u(-\infty) = B$, the solution corresponds to a strong detonation wave (see [7] [5]).

REMARK 1.1.

A solution (u,v) of (1.11) is only determined up to a translation of the origin. In particular, uniqueness results which are stated below are to be understood up to translations of the origin. ■

We can now state our main results :

THEOREM 1.2.

Consider the combustion profiles for equations (1.11).

Let $K > 0$, $d \geq 0$ and $q_0 \geq \hat{q}_0$ be fixed.

There is a critical value q^{CR}, with $q^{CR} \geq \hat{q}_0$, such that :

(i) For $q_0 = q^{CR}$, a unique weak detonation profile exists. On this profile, u is a monotone decreasing function of y.

(ii) For $q_0 > q^{CR}$, a unique strong detonation profile exists.

(iii) When $q^{CR} > \hat{q}_0$, for energy values q_0 with $\hat{q}_0 \leq q_0 < q^{CR}$, no combustion profile exists. ■

THEOREM 1.3.

Let r be defined by : $dr^2 + sr - K = 0$, $r > 0$. If q_0 satisfies

$$q_0 > \left(1 + \frac{dr}{s}\right) \left[\hat{q}_0 + \frac{rB}{s} + 2\left(\frac{\hat{q}_0 rB}{s}\right)^{1/2}\right],$$

then $q_0 > q^{CR}$ and the strong detonation profile from (ii) above has a nonmonotone spike in the u-profile : the function u has a unique maximum, and this maximum exceeds B. ■

THEOREM 1.4.

The following estimate holds, for fixed A, B, C_0 and s, and for $q^{CR} = q^{CR}(K,d)$:

$$(1.12) \qquad q^{CR}(K,d) \le (1 + \frac{dr}{s}) \quad [\hat{q}_0 + \frac{rA}{s} + 2(\frac{\hat{q}_0 rA}{s})^{1/2}].$$

For fixed $d \geq 0$, the function $q^{CR}(\cdot,d)$ is continuous and monotone increasing on $\mathbb{R}_+^*$, with

$$(1.13) \qquad \lim_{K\to 0} q^{CR}(K,d) = \hat{q}_0 \quad , \qquad \lim_{K\to +\infty} q^{CR}(K,d) = +\infty.$$

If $K > 0$ satisfies $q^{CR}(K,d) > \hat{q}_0$, then the function $q^{CR}(\cdot,d)$ is strictly increasing on the interval $[K, +\infty[$. ■

REMARK 1.5.

The results of Theorems 1.2, 1.3, and the properties (1.12), (1.13) were proved by A. Majda [7] in the case $d = 0$. For the study of the function $q^{CR}(K,0)$ and other results concerning this case, see the Appendix below. ■

REMARK 1.6.

The results of Theorems 1.2 are almost unchanged if the equation (1.6) has a unique solution $A = B > 0$. In contrast, if (1.6) admits two solutions satisfying $A \le 0 < B$, R. Rosales and A. Majda [9] indicate that there is no longer a critical energy value, in the case $d = 0$: for any positive q_0, a unique combustion profile solution of (1.11) exists. We will see that this too remains true when $d > 0$.

The next result concerns the passage to the limit $d \to 0$. ■

THEOREM 1.7.

The notations are those of Theorem 1.4. K is now a fixed positive number.

(i) $$\lim_{d\to 0} q^{CR}(K,d) = q^{CR}(K,0)$$

(ii) Let $y_0 \in \mathbb{R}$ and $F \in]0,A[$. Let (u_d,v_d) be the weak detonation profile defined in Theorem 1.2 (i), with $u_d(y_0) = F$. Then :

$$u_d \to u_0 \text{ in } C^1(\mathbb{R}) \text{ and } v_d \to v_0 \text{ in } C^0(\mathbb{R}) \text{ as } d \to 0.$$

(iii) Assume that q_0 satisfies $q_0 > q^{CR}(K,0)$. Then for sufficiently small values of d, $q^{CR}(K,d) < q_0$. Let $y_0 \in \mathbb{R}$ and $F \in]\,U(q_0),B[$. Let (u_d,v_d) be the strong detonation profile defined in Theorem 1.2 (ii), with $u_d(y_0) = F$. Then

$$u_d \to u_0 \text{ in } C^1(\mathbb{R}) \text{ and } v_d \to v_0 \text{ in } C^0(\mathbb{R}) \text{ as } d \to 0. \blacksquare$$

To conclude this introduction, let us emphasize that the methods we employ here are different from the ones used by A. Majda [7] in the case d = 0. Our problem (1.11) can be written as an autonomous 3 X 3 system of first-order ordinary differential equations (see (3.8) below). But the standard theorems which are valid for first-order 2 X 2 systems, on which are based the arguments in [7], do not apply for this more general system. Nevertheless, many an idea of [7] will still be used in the analysis below.

To prove the existence of solutions, we proceed as follows. First, we study an associated problem posed on a bounded domain $[-\alpha,\alpha]$, which approximates (1.11). This problem is solved by using the topological Leray-Schauder degree.

Then, with appropriate a priori estimates we take the limit of a solution of the associated problem as $\alpha \to +\infty$, and obtain an existence result for (1.11), namely Theorem 2.7 below. This limiting procedure along bounded intervals has some interest in its own right, especially in view of numerical computations[(1)].

In remark 3.12, we sketch another proof of the same existence result. This second method uses a shooting argument, similar to the one used by H. Berestycki, P.L. Lions and L.A. Peletier [2], or by H. Berestycki, B. Nicolaenko and B. Scheurer [3] and M. Marion [8] in other works concerning combustion waves.

The paper is organized as follows :

Section 2 : Associated problem - Strong detonation profiles.

Section 3 : Critical energy - Weak detonation profiles.

Section 4 : Continuous dependence - Asymptotic analysis.

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(1) We hope to publish our results about the numerical approximation of the combustion profiles in a forthcoming paper.

SECTION II : ASSOCIATED PROBLEM
STRONG DETONATION PROFILES

From now on, we assume $d > 0$.

For $\alpha \geq 1$, we denote by $I\alpha$ the closed interval $[-\alpha,\alpha]$ and by $X\alpha$ the Banach space $C^1(I\alpha) \times C^1(I\alpha)$. We consider the problem :

Find $(u,v) \in C^2(I\alpha) \times C^2(I\alpha)$, such that :

$$(2.1)\quad \begin{cases} u'' = -su' - sv' + a(u)u' , \\ dv'' + sv' = K\,\Phi(u)v , \\ u(-\alpha) = B, \quad u(0) = 0, \\ v(-\alpha) = 0, \; v(\alpha) = q_0 . \end{cases}$$

The reason for differentiating the first equation in (1.11) will become clear in the analysis below. About problem (2.1), we will prove :

PROPOSTION 2.1

For any positive value of q_0, there exists a solution (u,v) of problem (2.1). Moreover, there exists a constant R independant of $\alpha \in [1, +\infty[$, such that :

$$(2.2)\qquad \| (u,v) \|_{X\alpha} \leq R .$$

Proof : It relies on the topological Leray-Schauder degree. For $\tau \in [0,1]$, consider the problem :

Find $(u,v) \in C^2(I\alpha) \times C^2(I\alpha)$ such that :

$$(2.3)\quad \begin{cases} u'' = \tau(-\,su' - sv' + a(u)u'), \\ dv'' + sv' = \tau K\phi(u)v, \\ u(-\alpha) = B, \quad u(0) = 0, \\ v(-\alpha) = 0, \quad v(\alpha) = q_0. \end{cases}$$

For $\tau \in [0,1]$, we can also define a mapping :

$$\mathcal{F}_\tau : \begin{cases} X\alpha \to X\alpha \\ (u,v) \to \mathcal{F}_\tau(u,v) = (U,V), \text{ with :} \end{cases}$$

$$\begin{cases} U'' = \tau(-\,su' - sv' + a(u)u'), \\ dV'' + sV' = \tau K\phi(u)v, \\ U(-\alpha) = B, \quad U(0) = 0, \\ V(-\alpha) = 0, \quad V(\alpha) = q_0. \end{cases}$$

It is fairly classical that $\mathcal{F}_\tau$ is a continuous and compact operator from $X\alpha$ to $X\alpha$. Furthermore, the mapping :

$$H : \begin{cases} X_\alpha \times [0,1] \to X_\alpha \\ [(u,v)\ ;\ \tau] \to \mathcal{F}_\tau(u,v) \end{cases}$$

is compact and uniformly continuous with respect to τ. Notice that problems (2.1) and (2.3) are equivalent to $\mathcal{F}_1(X) = X$ and $\mathcal{F}_\tau(X) = X$ respectively. Set $G_\tau = Id_{X\alpha} - \mathcal{F}_\tau$. We want to solve (2.3), or to find $(u,v) \in X\alpha$ such that $G_\tau(u,v) = (0,0)$: we are going to show that the degree of G_τ about 0 is well defined and different from zero.

The next lemmas provide us with estimates which are required to compute the degree :

LEMMA 2.2.

Let $\tau \in [0,1]$. Let (u,v) be a solution of (2.3). Then :

(2.4) $\quad \forall\, y \in I\alpha, \quad 0 \leq v(y) \leq q_0,$

(2.5) $\quad \forall\, y \in I\alpha, \quad v'(y) > 0.$

Proof : a) Set $h(y) = \tau K\phi(u(y)) \geq 0$. (2.4) is a straightforward consequence of the maximum principle for the linear elliptic equation $-dv'' - sv' + h(y)v = 0$, together with $v(\partial I\alpha) = \{0,q_0\}$.

b) From (2.4), we get : $v'(-\alpha) \geq 0$. If $v'(-\alpha) = v(-\alpha) = 0$, the differential equation for v implies : $v \equiv 0$. Therefore, we have : $v'(-\alpha) > 0$.

Furthermore, we can write, for $y \in I\alpha$:

$$(v'(y)\, e^{\frac{sy}{d}})' = \frac{h(y)}{d}\, e^{\frac{sy}{d}} \geq 0,$$

whence : $\quad v'(y)\, e^{\frac{sy}{d}} \geq v'(-\alpha)\, e^{\frac{-s\alpha}{d}} > 0$, and the proof is complete. ■

Let $\tau \in\,]0, 1]$. If (u,v) is a solution of (2.3), (2.5) yields :

$$\forall\, y \in I\alpha,\ u'(y) = 0 \Rightarrow u''(y) = -\tau s v'(y) < 0$$

which obviously implies :

COROLLARY 2.3 :

Let $\tau \in]0,1]$. Let (u,v) be a solution of (2.3). The following alternative holds :

$$(2.6)\quad \begin{cases} \text{Either} : & \forall\, y \in I\alpha,\ u'(y) < 0\ ; \\ \text{Or} \quad : & \exists\, y_\alpha \in I\alpha,\ \forall\, y < y_\alpha,\ u'(y) > 0, \\ & \qquad \forall\, y > y_\alpha,\ u'(y) < 0. \blacksquare \end{cases}$$

LEMMA 2.4 (Upper bound for v').

Let $\tau \in [0,1]$. Let (u,v) be a solution of (2.3). Then :

$$\forall\, y \in I\alpha,\ v'(y) \leq \max\left(\frac{\tau K q_0}{s}, \frac{q_0 s}{d} + \frac{q_0}{2\alpha}\right).$$

Proof : Define $y_0 \in I\alpha$ with $v'(y_0) = \max\limits_{I\alpha} v'(y)$. If $y_0 \in \mathring{I}\alpha$, we have : $v''(y_0) = 0$. If $y_0 = \alpha$, we get : $v''(y_0) \geq 0$. In both cases, we obtain : $sv'(y_0) \leq \tau K\, v(y_0)\, \phi\, (u(y_0))$, whence $sv'(y_0) \leq \tau K q_0$.

It remains to study the case $y_0 = -\alpha$. For $y \in I\alpha$, we have : $dv''(y) + sv'(y) \geq 0$. Integrating this inequality between $-\alpha$ and y, we get : $d(v'(y) - v'(-\alpha)) + sv(y) \geq 0$. But we can choose $y \in I\alpha$ such that $v'(y) = \dfrac{v(\alpha) - v(-\alpha)}{2\alpha} = \dfrac{q_0}{2\alpha}$, and we obtain :

$$v'(-\alpha) = v'(y_0) \leq \frac{q_0}{2\alpha} + \frac{sq_0}{d}, \text{ which completes the proof. } \blacksquare$$

LEMMA 2.5 (Estimates for u and u')

Let $\tau \in]0,1]$. Let (u,v) be a solution of (2.3). There exists a constant R_1 independent of $\tau \in]0,1]$ and $\alpha \in [1, +\infty[$, such that :

$$\forall\, y \in I\alpha, \quad |u(y)| < R_1\ ,\ |u'(y)| < R_1.$$

REMARK 2.6

The independence of the estimates with respect to α does not play any role to prove the existence result stated in Proposition 2.1. But il will be useful later on, to take the limit $\alpha \to +\infty$. ■

Proof : Obviously there exists a constant $C\alpha$ such that :

$$\begin{cases} u' = \tau(-su - sv + f(u) + C\alpha) \\ dv'' + sv' = \tau K\phi(u)v \end{cases}$$

a) Upper bound for $C\alpha$

Using (1.3) and (1.9), we have :

$\forall\, u < A,\ a(u) - s < a(A) - s < 0,$

whence : $\forall\, u < 0,\ f(u) - su > f(0).$

Since (2.6) yields $u(\alpha) < u(0) = 0$ and $u'(\alpha) < 0$, we get :

(2.7) $\quad f(0) - sq_0 + C\alpha < -su(\alpha) + f(u(\alpha)) - sq_0 + C\alpha < 0,$

and therefore : $C\alpha < sq_0 - f(0)$.

b) Lower bound for $C\alpha$

For $(p,q) \in \mathbb{R}^2$, we define : $g_0(p,q) = -sp - sq + f(p) + C_0$, and $g_\alpha(p,q) = -sp - sq + f(p) + C\alpha$. (Notice that $u'(-\alpha) = \tau\, g_\alpha(B,0)$ and $g_0(B,0) = 0$).

Assume that $C\alpha < C_0$. Then : $u'(-\alpha) < 0$, and u is strictly decreasing, whence : $\forall\, y \in [-\alpha,0]$, $u(y) \in [0,B]$. But the hypothesis on f and s imply :

$\forall\, u \in [0,B],\ f(u) - su \leq f(0)$ (see Fig. 1).

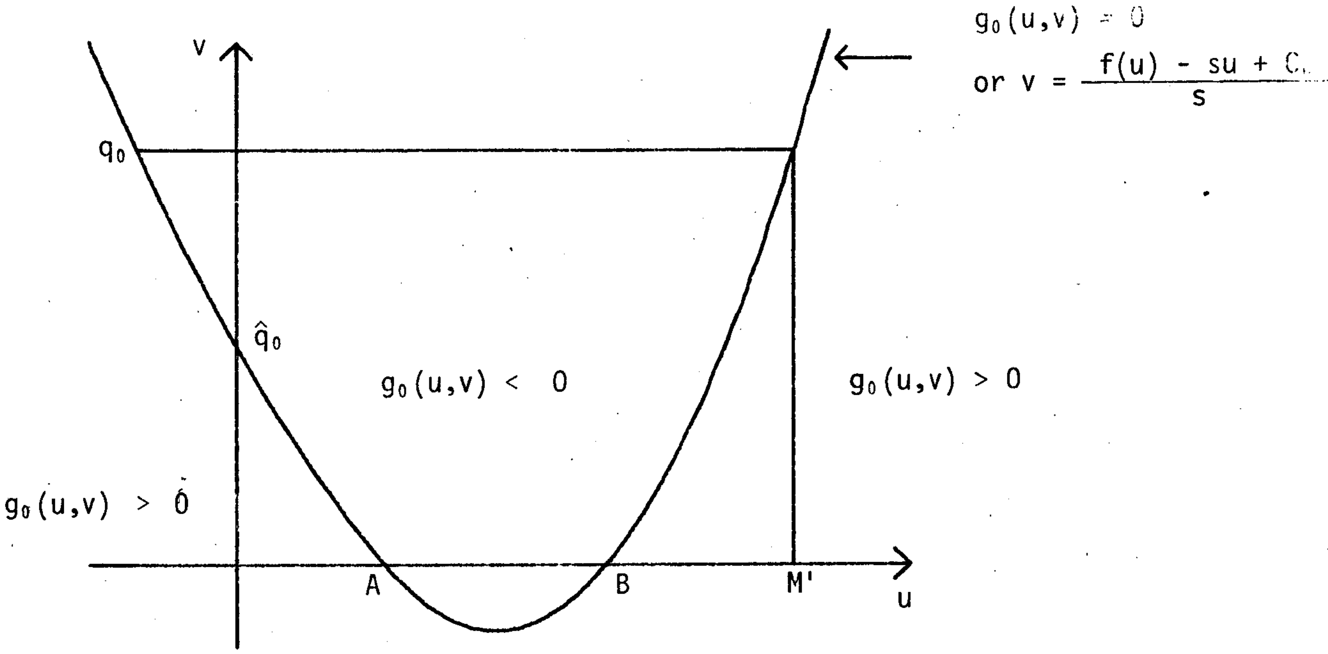
Then : $\forall\, y \in [-\alpha,0]$, $u'(y) \leq \tau\,[f(0) + C\alpha - sv(y)] \leq \tau\,[f(0) + C\alpha]$.

Integrating the last inequality between $-\alpha$ and 0, we get :

(2.8) $\quad -B \leq \tau\alpha(f(0) + C\alpha)$,

or $\quad C\alpha \geq \dfrac{-B}{\tau\alpha} - f(0)$.

Figure 1



If we now assume $C\alpha \geq C_0$, we can write :

$C\alpha \geq C_0 > -f(0) > \dfrac{-B}{\tau\alpha} - f(0)$, where we used (1.8) and (1.10).

We have therefore proved the following estimate :

(2.9) $\quad \dfrac{-B}{\tau\alpha} - f(0) \leq C\alpha \leq sq_0 - f(0)$.

c) Lower bound for u

We have : $\forall\, y \in [0,\alpha]$, $u(y) \leq 0$, whence :

$\forall\, y \in [0,\alpha]\ u'(y) \geq \tau\,[f(0) + C\alpha - sv(y)] \geq \tau\,[f(0) + C\alpha - sq_0]$.

We integrate the last inequality between 0 and α, and use (2.8) to get :

$u(\alpha) \geq -B - sq_0\,\tau\alpha$.

On the other hand, we have from (2.7) :

$- su(\alpha) + f(u(\alpha)) - sq_0 + C\alpha < 0$, which implies :

$- su(\alpha) + f(0) + a(0)u(\alpha) - sq_0 + C_\alpha < 0$, because of the convexity of f.

Thus : $u(\alpha) > (C\alpha + f(0) - sq_0) . [s - a(0)]^{-1}$.

Finally, we obtain :

$\forall y \in I\alpha,\ u(y) \geq u(\alpha) \geq \max\left\{-B - sq_0\,\tau\alpha\ ;\ (\frac{-B}{\tau\alpha} - sq_0) . (s - a(0))^{-1}\right.$.

It is then obvious that there exists a constant M, independent of $\tau \in]0,1]$ and $\alpha \in [1,+\infty[$, such that :

$\forall y \in I\alpha,\ u(y) \geq M.$

d) Upper bound for u

If $C\alpha < C_0$, u is decreasing : $\forall y \in I\alpha$, $u(y) \leq B$. Let us then assume $C\alpha \geq C_0$, and $\max_I u(y) = u(y_\alpha) > B$. Then :

$u'(y_\alpha) = \tau g_\alpha [u(y_\alpha),v(y_\alpha)] = 0$, which yields : $g_\alpha [u(y_\alpha), q_0] \leq 0$,

and $g_0 [u(y_\alpha), q_0] \leq 0$. The point $[u(y_\alpha), q_0]$ lies in the region $\{g_0 \leq 0\}$.

Therefore $u(y_\alpha)$ is bounded (see Fig. 1), and

$\forall y \in I\alpha,\ u(y) \leq M'$,

where M' is a constant independent of τ and α.

e) Estimate for u'

From : $u' = \tau [- su - sv + f(u) + C\alpha]$ and 2.9, we easily deduce lower and upper bounds for u', independently of $\tau \in]0,1]$ and $\alpha \in [1, +\infty[$. ■

We can now complete the proof of proposition 2.1. Up to now, we have considered the case $\tau \neq 0$. But the case $\tau = 0$ is very simple, since the mapping G_0 is a translation. We can then extend the results of Lemmas 2.2 to 2.5 and state :

Let $\tau \in [0,1]$ and $\alpha \in [1, +\infty[$. Let (u,v) be a solution of (2.3). There exists a positive constant R independent of τ and α such that :

$$\| (u,v) \|_{X_\alpha} < R$$

We may therefore compute the degree $d\,[\mathcal{G}_\tau\,; \mathcal{B}_R; (0,0)]$ where $\mathcal{B}_R = \{(u,v) \in X\alpha,\ \|(u,v)\|_{X\alpha} < R\}$. The homotopy invariance of the Leray-Schauder degree yields :

$\forall\, \tau \in [0,1]$, $d\,[\mathcal{G}_\tau\,; \mathcal{B}_R; (0,0)] = d\,[\mathcal{G}_0\,; \mathcal{B}_R; (0,0)]$. This last degree is obviously equal to 1, and $d\,[\mathcal{G}_1; \mathcal{B}_R; (0,0)] = 1$: there exists a solution to problem (2.1), and (2.2) holds. ■

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Now, we want to prove the following existence result :

THEOREM 2.7

There exists a value $q^c > q_0$, such that :

If the energy q_0 satisfies $q_0 > q^c$, then the problem (1.11) has a solution (u,v). Furthermore, this solution is a strong detonation profile : $u(-\infty) = B$.

The proof of this theorem is divided into a sequence of lemmas or propositions. The next statement follows from Proposition 2.1 :

LEMMA 2.8

For $\alpha \in [1, +\infty]$, define $(u_\alpha, v_\alpha) \in X\alpha$ as a solution of (2.1). There exists an increasing sequence (α_n) with $\alpha_n \geq 1$ and $\lim \alpha_n = +\infty$, such that the sequence $(u_{\alpha_n}, v_{\alpha_n})$ converges in $C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R})$ to (u,v) satisfying :

$$\begin{cases} u'' = - su' - sv' + a(u)\, u', \\ dv'' + sv' = K\phi(u)\, v. \blacksquare \end{cases}$$

In the sequel, we will forget the integer n, and still denote (u_α, v_α) the converging sequence. Notice that we have obviously : $(u,v) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$, and

(2.10) $\quad u' = - su - sv + f(u) + C,$

with $C = \lim C\alpha$.

LEMMA 2.9

Let (u,v) be defined by Lemma 2.8. Then :

(2.11) $\qquad - \infty < u\,(+\infty) \leq 0 \quad , \quad v(+\infty) = q_0.$

$$(2.12) \begin{cases} \text{Either} & \forall\, y \in \mathbb{R},\ u'(y) \leq 0, \\ \text{Or} & \exists\, y_0 \leq 0,\ \forall\, y > y_0,\ u'(y) \leq 0, \\ & \qquad \forall\, y < y_0,\ u'(y) \geq 0 \text{ and } u(y) \geq B. \end{cases}$$

$$(2.13) \begin{cases} \text{Either} & u \equiv 0\ ,\ v \equiv q_0, \\ \text{Or} & 0 < u(-\infty) < + \infty\ ,\ v(-\infty) = 0. \end{cases}$$

Proof a) Proof of (2.11) : For $\alpha \in [1,+\infty[$ and $y \in [0,\alpha]$, we have : $u_\alpha(y) \leq 0$, $dv''_\alpha(y) + sv'_\alpha(y) = 0$. Thus :

$$v_\alpha(y) = q_0 - \frac{d}{s}\, v'_\alpha(0)\, \left(e^{-\frac{sy}{d}} - e^{-\frac{s\alpha}{d}}\right).$$

Taking the limit $\alpha \to + \infty$, we get :

(2.14) $\quad \forall\, y \in \mathbb{R}+ \quad , \quad v(y) = q_0 - \frac{d}{s} v'(0)\, e^{\frac{-sy}{d}}$, and $v(+\infty) = q_0$.

Furthermore : $\forall\, y \in \mathbb{R}+$, $u'(y) = \lim_{\alpha \to +\infty} u'_\alpha(y) \leq 0$; u is a monotone decreasing function on $\mathbb{R}+$. Since u is bounded, we have : $-\infty < u(+\infty) \leq 0 = u(0)$.

b) Proof of (2.12) : Let us assume :

$$\exists\, y \in \mathbb{R} \text{ , } u'(y) > 0. \tag{2.15}$$

Then, for sufficiently large values of α, $u'_\alpha(y)$ is strictly positive too : u_α is nonmonotone and has a maximum $y_\alpha \in \,]y,0[$. The sequence y_α is bounded ; we claim that it is convergent. To see this, consider two subsequences y_{α_1} and y_{α_2} with $\lim_{\alpha_1 \to \infty} y_{\alpha_1} = y_1 < y_2 = \lim_{\alpha_2 \to \infty} y_{\alpha_2}$. For $y \in \,]y_1,y_2[$, we have :
$u'(y) = \lim_{\alpha_1 \to \infty} u'_{\alpha_1}(y) \leq 0$ and $u'(y) = \lim_{\alpha_2 \to \infty} u'_{\alpha_2}(y) \geq 0$, whence $\forall\, y \in \,]y_1,y_2[$, $u'(y) = 0$: u is constant on $]y_1,y_2[$. From (2.10), v and therefore v' are constant on this interval. Then u and v are constant on $\mathbb{R}$, which contradicts (2.15).

Therefore we have : $\lim_{\alpha \to +\infty} y_\alpha = y_0$, and (2.12) is proved.

c) Proof of (2.13) : Assume that u is not constant. Then, (2.12) implies the existence of $u(-\infty)$, with $0 < u(-\infty)$. Since v is obviously an increasing function, $v(-\infty)$ exists in $\mathbb{R}$.
Let $\ell = K\phi\,[u(-\infty)].v(-\infty) = \lim_{y \to -\infty} [dv''(y) + sv'(y)]$. It is then easy to show that $\lim_{y \to -\infty} sv'(y) = \ell$, which yields : $\ell = 0$, whence $v(-\infty) = 0$.

To complete the proof, notice that, if u is constant, $u \equiv u(0) = 0$. v is constant from (2.10), and $v \equiv v(+\infty) = q_0$. ■

We can now show a more precise result about $u(-\infty)$:

LEMMA 2.10

If u is not constant, $u(-\infty) \in \,]0,A] \cup \{B\}$.

<u>Proof</u> : For $(p,q) \in \mathbb{R}^2$, we define : $g(p,q) = - sp - sq + f(p) + C$.

$(C = \lim_{\alpha \to +\infty} C\alpha)$.

a) Assume first that $C < C_0$. Then $C\alpha < C_0$ for large values of α, and u_α is decreasing : $\forall\, y \in I\alpha$, $u_\alpha(y) \le B$. Therefore : $\forall\, y \in \mathbb{R}$, $u(y) \le B$, and $u(-\infty) \le B$.

Moreover, we know that : $g\,[u(-\infty),0] = 0$, from (2.10). Then : $g_0\,[u(-\infty),0] = g\,[u(-\infty),0] + C_0 - C > 0$. The point $[u(-\infty),0]$ lies in the region $g_0 > 0$. Together with $u(-\infty) \le B$, this implies $u(-\infty) < A$ (see Fig. 1).

b) Let us now suppose $C > C_0$. Arguing as above, we see that u_α is non-monotone for large values of α : Let $\varepsilon \in \,]0,C-C_0[$. Then :

$$\exists\, \alpha_0 \ge 1 \;,\; \forall\, \alpha \ge \alpha_0 \quad,\quad C\alpha > C - \varepsilon > C_0 .$$

For $\alpha \ge \alpha_0$, we can write :

$$\exists\, y_\alpha \in I\alpha \quad,\quad u_\alpha'(y_\alpha) = 0 \quad,\quad u_\alpha(y_\alpha) \ge B.$$

Hence : $\quad u_\alpha'(y_\alpha) = 0 = g_\alpha\,[u_\alpha(y_\alpha),v_\alpha(y_\alpha)] > - sB + f(B) - sv_\alpha(y_\alpha) + C - \varepsilon$

which yields :

$$v_\alpha(y_\alpha) > \frac{C - \varepsilon - sB + f(B)}{s} = \frac{C - \varepsilon - C_0}{s} = \varepsilon' > 0.$$

From (2.5) and (2.6), we have :

$$\forall\, y \in I\alpha \;,\; u_\alpha(y) < B \Rightarrow y > y_\alpha \Rightarrow v_\alpha(y) > v_\alpha(y_\alpha) > \varepsilon' .$$

Taking the limit $\alpha \to +\infty$, we get :

$$\forall\, y \in \mathbb{R},\; u(y) < B \Rightarrow v(y) > \varepsilon' > 0.$$

If u is not constant, we have $v(-\infty) = 0$, whence $u(-\infty) \ge B$. But we know that : $g\,[u(-\infty),0] = 0$; since $C > C_0$, the point $[u(-\infty),0]$ lies in the region $\{g_0 < 0\}$, which contradicts $u(-\infty) \ge B$ (see Fig. 1). The case $C > C_0$ is therefore impossible if u is not constant.

c) Since the case $C = C_0$ gives obviously $u(-\infty) \in \{A,B\}$, the proof is achieved. ■

Now , we want to prove that u is not constant if the energy q_0 is sufficiently large. We first need to define some notations. For $\varepsilon \geq 0$, let B'_ε be the solution of : $B'_\varepsilon > 0$ and $f(B'_\varepsilon) - sB'_\varepsilon = f(0) + \varepsilon$. We have $B'_\varepsilon > B$ (see fig. 1), and we denote $B'_\varepsilon = B + b_\varepsilon$.

LEMMA 2.11

Let $\tilde{q}$ be defined by : $\tilde{q} > \hat{q}_0$ and $\frac{d}{4b_0}(\tilde{q} - \hat{q}_0)^2 - \tilde{q} = 0$.

Then, if q_0 satisfies $q_0 > \tilde{q}$, u is not constant.

Proof : If u is constant, we have $u \equiv 0$, $v \equiv q_0$, and, from (2.10), $C = sq_0 - f(0)$. Therefore it suffices to show that the equality $C = sq_0 - f(0)$ implies $q_0 \leq \tilde{q}$.

Let $q_0 > \hat{q}_0$, and assume that $C = \lim\limits_{\alpha \to +\infty} C\alpha = sq_0 - f(0)$. We recall that $C_0 = sB - f(B) = s\hat{q}_0 - f(0)$. Then : $C = C_0 + s(q_0 - \hat{q}_0) > C_0$. For $\alpha \in [1,+\infty[$, we have, from (2.9) : $C\alpha \leq C$. Let us define : $\varepsilon = C - C\alpha \geq 0$; $v_\varepsilon = \frac{-sB + f(B) + C\alpha}{2s} = \frac{C\alpha - C_0}{2s}$. (Notice that ε depends on α !).

For large values of α, we have : $C\alpha > C_0$, and u_α has a maximum : $u_\alpha(y_\alpha) > B$. Then, we can write :

$$- su_\alpha(y_\alpha) + f(u_\alpha(y_\alpha)) - sv_\alpha(y_\alpha) + C\alpha = 0,$$

$$- su_\alpha(y_\alpha) + f(u_\alpha(y_\alpha)) = sv_\alpha(y_\alpha) - C_\alpha \leq sq_0 - C_\alpha = f(0) + C - C_\alpha = - sB'_\varepsilon + f(B'_\varepsilon),$$

whence $u_\alpha(y_\alpha) \leq B'_\varepsilon$.

Furthermore, we have :

$$sv_\alpha(y_\alpha) = - su_\alpha(y_\alpha) + f(u_\alpha(y_\alpha)) + C\alpha \geq - sB + f(B) + C\alpha = 2s\, v_\varepsilon,$$

which implies : $v_\alpha(y_\alpha) \geq 2v_\varepsilon$.

Now, we take $y_0 \in [-\alpha, y_\alpha]$ such that $v_\alpha(y_0) = v_\varepsilon$. For $y \in [-\alpha, y_0]$, we have :

$$v(y) \leq v_\varepsilon \text{ and } B \leq u_\alpha(y) \leq B'_\varepsilon = B + b_\varepsilon ;$$

thus : $\quad u'_\alpha(y) \geq - sB + f(B) - sv_\varepsilon + C\alpha = sv_\varepsilon$.

After integration, we get :

$$b_\varepsilon \geq u_\alpha(y_0) - u_\alpha(-\alpha) \geq sv_\varepsilon(y_0 + \alpha),$$

or
$$y_0 + \alpha \leq \frac{b_\varepsilon}{sv_\varepsilon} .$$

Let y_1 $[-\alpha, y_0]$, such that :

$$v'_\alpha(y_1) = \frac{v_\alpha(y_0) - v_\alpha(-\alpha)}{y_0 + \alpha} \geq \frac{sv_\varepsilon^2}{b_\varepsilon} .$$

We know that : $\forall\, y \in I\alpha$, $(v'_\alpha(y)\, e^{\frac{sy}{d}})' \geq 0$.

Then, for $y \geq y_1$

$$v'_\alpha(y) \geq v'_\alpha(y_1)\, e^{\frac{sy_1}{d}} \cdot e^{\frac{-sy}{d}} \geq \frac{sv_\varepsilon^2}{b_\varepsilon}\, e^{\frac{sy_1}{d}}\, e^{\frac{-sy}{d}}$$

whence : $q_0 \geq v_\alpha(y_1 + \alpha) \geq \frac{dv_\varepsilon^2}{b_\varepsilon}(1 - e^{\frac{-s\alpha}{d}})$.

Taking the limit $\alpha \to +\infty (\varepsilon \to 0)$, we obtain :

$$q_0 \geq \frac{d}{b_0}\left(\frac{C - C_0}{2s}\right)^2 = \frac{d}{4b_0}(q_0 - \hat{q}_0)^2,$$ which obviously implies $q_0 \leq \tilde{q}$,

and the proof is complete. ■

The next proposition gathers all the previous results :

<u>PROPOSITION 2.12</u>

Consider the problem :

Find $(u,v) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ and $C \in \mathbb{R}$, such that :

$$(2.16)\quad \begin{cases} u' = -su - sv + f(u) + C, \\ dv'' + sv' = K\phi(u)v\ ; \\ v(-\infty) = 0\ ;\ v(+\infty) = q_0\ ;\ u(0) = 0\ ; \\ u(-\infty) \in\,]0,A] \cup \{B\}\ ;\ u(\mathbb{R}+) \subset \mathbb{R}_-\ ;\ -\infty < u(+\infty) < 0 \end{cases}$$

If the energy q_0 satisfies $q_0 > \tilde{q}$, (2.16) has a solution. ■

We can now explain why we differentiated the first equation of (1.11), getting two second order equations for problem (2.1), and of course two conditions on u_α. A unique condition like $u_\alpha(-\alpha) = B$ could have been adequate to ensure the boundedness and hence the existence of u_α. But the second condition $u_\alpha(0) = 0$ played a crucial role in obtaining the explicit expression (2.14) and the limit $v(+\infty)$. Naturally, it remains now to prove that we really solved the original problem (1.11), or equivalently that $C = C_0$ (or $u(-\infty) = B$).

In fact, we are going to show that $u(-\infty) = B$ when q_0 is sufficiently large. From Lemma 3.4, a natural idea is to find a function which acts as a barrier for (u,v) and finally prevents $u(-\infty)$ to be $\leq A$. The precise result is the following :

PROPOSITION 2.13

Let r be defined by : $dr^2 + sr - K = 0$, $r > 0$. For $q_0 > \tilde{q}$, let (u,v,C) be a solution of (2.16). If moreover :

(2.17) $$q_0 > (1 + \frac{dr}{s}) \{ \hat{q}_0 + \frac{rA}{s} + 2(\frac{\hat{q}_0 \, rA}{s})^{1/2} \}$$

then $u(-\infty) = B$ and (u,v) is a solution of problem (1.11).

We just require three lemmas :

LEMMA 2.14

For $q_0 > \tilde{q}$, let (u,v,C) be a solution of (2.16). Then :

(2.18) $$v(0) \geq \frac{s}{dr + s} q_0 .$$

Proof : Let $\tilde{v}(y) = v(0)\, e^{ry}$ for $y \leq 0$. We have : $\tilde{v}(0) = v(0)$; $\tilde{v}(-\infty) = v(-\infty) = 0$, and $d\tilde{v}'' + s\tilde{v}' - K\tilde{v} = 0$, whereas : $d v'' + sv' - Kv = K(\Phi(u) - 1)v \leq 0$

Using the maximum principle for elliptic equations, on decuces easily that , these properties yield : $\forall\ y \in \mathbb{R}-$, $v(y) \geq \tilde{v}(y)$. Since $v(0) = \tilde{v}(0)$, we get : $v'(0) \leq \tilde{v}'(0) = rv(0)$. But $v'(0) = \frac{s}{d}(q_0 - v(0))$ from (2.14), and (2.18) is proved. ∎

The next lemma is due to A. Majda [7]. We will just indicate the main arguments of the proof. Consider the following autonomous system :

$$(2.19)\quad \begin{cases} \hat{Q}' = -s\hat{Q} - s\hat{V} + f(\hat{Q}) + C, \\ \hat{V}' = r\hat{V} \end{cases}$$

LEMMA 2.15

Let $a \in]0,B]$. Denote $\hat{q} = \dfrac{f(0) + C}{s}$, and let $\hat{V}_0$ $]\hat{q},+\infty[$. Consider the integral curve $\hat{C}$ of (2.19) through the point $(0,\hat{V}_0)$. $\hat{V}$ is a monotone decreasing function of $\hat{Q}$ along $\hat{C}$ as long as this curve remains in the region : $P = \{\ (\hat{Q},\hat{V}),\ 0 < \hat{Q} \leq a,\ -s\hat{Q} - s\hat{V} + f(\hat{Q}) + C < 0\}$. Denote $[\hat{Q},\ \hat{V}(\hat{Q})]$ this piece of curve. If $\hat{V}_0$ satisfies :

$$(2.20)\qquad \hat{V}_0 \geq \hat{q} + \frac{ra}{s} + 2\left(\frac{\hat{q}ra}{s}\right)^{1/2}$$

then $[\hat{Q},\ \hat{V}(\hat{Q})] \in P$ and $\hat{V}(\hat{Q}) > \hat{q}$ for $\hat{Q} \in [0,a]$.

Proof : The monotonicity of $\hat{V}(\hat{Q})$ as long as the curve remains in P is straightforward (see [7]). We now prove the second part of the lemma. Since $a \leq B$ we deduce from (1.3) that $\max\limits_{\hat{Q} \in [0,a]} \left(\dfrac{f(\hat{Q}) + C - s\hat{Q}}{s}\right) = \hat{q}$ (see Fig. 1). Assume now that, along the integral curve, we have :

$$(2.21)\qquad \hat{V}(\hat{Q}) \geq (1 + b)\hat{q}$$

for $\hat{Q} \in [0,a]$ and for some positive b. Then $[\hat{Q},\hat{V}(\hat{Q})] \in P$, and : $\left|\frac{d\hat{v}}{d\hat{u}}\right| = \frac{r\hat{v}}{s\left[\hat{v} - \frac{f(\hat{u}) + C - s\hat{u}}{s}\right]} \leq \frac{r\hat{v}}{s(\hat{v}-\hat{q})} \leq \frac{r}{s}\ \frac{1+b}{b}$.

By integration, we get : $\hat{v}(a) \geq \hat{v}_0 - a\,\frac{r}{s}\ \frac{1+b}{b}$. As $\hat{v}$ is a decreasing function of $\hat{u}$, it suffices to choose $\hat{v}_0$ such that :

$\hat{v}_0 \geq (1+b)\hat{q} + a\,\frac{r}{s}\ \frac{1+b}{b}$ to ensure that our assumption (2.21) is fulfilled. It remains now to minimize the right-hand side of the last inequality over $b > 0$ to get (2.20). The proof is then easily finished. ■

Denoting $\mathcal{C} = \{\ [u(y), v(y)]\ ,\ y \leq 0\}$, we have :

LEMMA 2.16

If $\hat{v}_0 < v(0)$, the curve $\mathcal{C}$ remains above the curve $\hat{\mathcal{C}}$ as long as both curves stay in the region P.

Proof : We define an origin on the curve $\hat{\mathcal{C}}$ by setting : $\hat{u}(0) = 0$, $\hat{v}(0) = \hat{v}_0$. Let $\mathcal{D} = \{\ y \in \mathbb{R}_-,\ [\hat{u}(y), \hat{v}(y)] \in P\ \}$.

We already know that : $\forall\ y \in \mathcal{D}\ ,\ v(y) \geq v(0)\ e^{ry} > \hat{v}_0\ e^{ry} = \hat{v}(y)$:

We also have $u'(0) = -\,sv(0) + f(0) + C < \hat{u}'(0)$. Since $u(0) = \hat{u}(0)$, we get :

$\exists\ \varepsilon > 0,\ \forall\ y \in\ ]-\varepsilon, 0[\,,\ u(y) > \hat{u}(y)$. Let us assume :

$\exists\ y_0 \in \mathcal{D},\ \ \forall\ y \in\ ]y_0, 0[\,,\ u(y) > \hat{u}(y)$ and $u(y_0) = \hat{u}(y_0)$.

Then : $u'(y_0) - \hat{u}'(y_0) = -\,sv(y_0) + s\hat{v}(y_0) < 0$, which is impossible.

Therefore : $\forall\ y \in \mathcal{D}\ ,\ u(y) > \hat{u}(y),\ v(y) > \hat{v}(y)$, and the proof is complete. ■

We can achieve the proof of Proposition 2.13. If $q_0 > \tilde{q}$ satisfies (2.17), we get, from (2.18) :

$v(0) > \hat{q}_0 + \frac{rA}{s} + 2\left(\frac{\hat{q}_0 rA}{s}\right)^{1/2}$. Let $\hat{v}_0 \in \left[\hat{q}_0 + \frac{rA}{s} + 2\left(\frac{\hat{q}_0 rA}{s}\right)^{1/2},\ v(0)\right[$.

Since $C \leq C_0$ and $\hat{q} \leq \hat{q}_0$, $\hat{v}_0$ satisfies (2.20) for $a = A$. From Lemmas 2.15 and 2.16, we know that the curve $\mathcal{C}$ does not hit the segment $\{\ 0 \leq u \leq A,\ v = \hat{q}\ \}$.

Therefore, we have : $u(-\infty) > A$, whence $u(-\infty) = B$ and $C = C_0$. ■

We have then proved the existence result of Theorem 2.7. q^c is defined by :

$$q^c = \max \left\{ \tilde{q} \; ; \; (1 + \frac{dr}{s}) \, [\hat{q}_0 + \frac{rA}{s} + 2(\frac{\hat{q}_0 rA}{s})^{1/2}] \right\}.$$

The next statement is an obvious consequence of this section, together with our uniqueness result (Theorem 3.3) below.

THEOREM 2.17

For $\alpha \in [1, +\infty[$, let (u_α, v_α) be a solution of the associated problem (2.1). If q_0 satisfies $q_0 > q^c$, the whole sequence (u_α, v_α) is converging in $C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R})$ to the unique solution of the problem (1.11) as $\alpha \to +\infty$. ■

SECTION III : CRITICAL ENERGY
WEAK DETONATION PROFILES

In this section, we will state a uniqueness result and characterize the set of values of the energy q_0 for which (1.11) admits a solution. We will also show the existence of a unique weak detonation profile.

From now on, we drop the subscript 0 in C_0 and $\hat{q}_0$ denoting only C and $\hat{q}$; C is now a fixed constant : $C = sB - f(B) = sA - f(A)$.

We first state two simple lemmas which will be used several times in the sequel. The proof of Lemma 3.1 is obvious and will be omitted.

LEMMA 3.1

Let (u,v) be a solution of (1.11). The following properties hold :

$$(3.1)\qquad \begin{cases} \text{Either :} & \forall\, y \in \mathbb{R},\ u'(y) < 0\ ; \\ \text{Or} & \exists!\, y_0 \in \mathbb{R},\ \forall\, y < y_0,\ u'(y) > 0,\ u(y) > B; \\ & \qquad \forall\, y > y_0,\ u'(y) < 0 \end{cases}$$

$$(3.2)\qquad \forall\, y \in \mathbb{R},\ v'(y) > 0.$$

$$(3.3)\qquad \exists!\, y_2 \in \mathbb{R},\ u(y_2) = U_0 \text{ and } \forall\, y < y_2,\ v'(y) = rv(y).$$

$$(3.4)\qquad \begin{cases} \text{If } q_0 > \hat{q},\ \exists!\, y_1 \in \mathbb{R},\ u(y_1) = 0, \\ \quad \text{and } \forall\, y > y_1,\ sv(y) + dv'(y) = sq_0 . \blacksquare \end{cases}$$

(We recall that U_0 and r are defined in (1.2) and Proposition 2.13 respectively).

LEMMA 3.2

For $i \in \{0,1\}$, let $(u_i, v_i) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ satisfying :

$$(3.5)\quad \begin{cases} u' = -\tilde{s}\,u - s\,v + f(u) + C, \\ dv'' + sv' = K\,\phi\,(u)v. \end{cases}$$

Moreover assume that : $\begin{cases} u_0\,(0) = u_1\,(0); v_0(0) - v_1(0) = S \geq 0\ ; \\ v'_1\,(0) - v'_0(0) = S' \geq 0; \quad S + S' > 0\,. \end{cases}$

Then :

$$(3.6)\begin{cases} \forall\ y \in \mathbb{R}_-^*,\ u_0(y) > u_1(y), \\ \forall\ y \in \mathbb{R}_-,\ v_0(y) \geq v_1(y) + S + \frac{d}{s} S'\ (\ e^{-\frac{sy}{d}} - 1). \end{cases}$$

Proof : a) The preceding assumptions imply :

$$(3.7)\qquad \exists\ \varepsilon > 0,\ \forall\ y \in\ ]-\varepsilon,0[,\ u_0(y) > u_1(y),\ v_0(y) > v_1(y).$$

If $S > 0$, we have indeed $v_0(0) > v_1(0)$ and $u'_0(0) < u'_1(0)$,

which gives (3.7). If $S=0$ and $S' > 0$, we have $u'_0(0) = u'_1(0)$, $u''_0(0) > u''_1(0)$ and $v'_1(0) < v'_0(0)$, and again (3.7).

b) Let $y_0 < 0$ such that :

$$\forall\ y \in\ ]y_0,0[,\ u_0(y) > u_1(y),\ v_0(y) > v_1(y).$$

Then, for $y \in]y_0,0[$:

$$(v_0'(y) e^{\frac{sy}{d}})' = \frac{K}{d}\,\Phi(u_0(y))v_0(y)\ e^{\frac{sy}{d}} \geq (v_1'(y)\ e^{\frac{sy}{d}}$$

Integrating between y and 0, we get :

$$v'_0(0) - v'_0(y)\ e^{\frac{sy}{d}} \geq v'_1(0) - v'_1(y)\ e^{\frac{sy}{d}}\ ,$$

or

$$v_1'(y) \geq v_0'(y) + S' e^{\frac{-sy}{d}}\,.$$

Integrating again : $v_0(y_0) \geq v_1(y_0) + S + \frac{dS'}{s}\ (e^{-\frac{sy_0}{d}} - 1)$

In particular, we have : $v_0(y_0) > v_1(y_0)$.

If $u_0(y_0) = u_1(y_0)$, we get : $u_0'(y_0) - u_1'(y_0) = - sv_0(y_0) + sv_1(y_0) < 0$, which is impossible. Therefore : $u_0(y_0) > u_1(y_0)$ and the proof is easily achieved. ■

The next theorem is now an easy consequence of the two previous lemmas :

THEOREM 3.3

If the energy q_0 satisfies $q_0 > \hat{q}$, there exists at most one solution to problem (1.11).

Proof : For $q_0 > \hat{q}$, let (u_0,v_0) and (u_1,v_1) be two different solutions of (1.11). Using (3.4), we define an origin on both curves (u_0,v_0) and (u_1,v_1) by setting : $u_0(0) = u_1(0) = 0$. If $v_0(0) = v_1(0)$, we have : $v_0'(0) = v_1'(0)$ from (3.4) and the two solutions coincide : $(u_0,v_0) \equiv (u_1,v_1)$. We may there-fore assume : $v_0(0) > v_1(0)$. The assumptions of Lemma 3.2 are then fulfilled and (3.6) yields : $v_0(-\infty) - v_1(-\infty) = +\infty$, whence a contradiction. ■

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We denote by A the set of values of $q_0 \geq \hat{q}$ such that (1.11) admits at least a solution. We also define $B \subset A$ as the set of q_0 for which a solution (u,v) of (1.11) is a strong detonation profile : $u(-\infty) = B$. We already know that A and B contain the interval $]q^{C}, +\infty[$. We are going to prove the next result :

THEOREM 3.4

A is a closed interval $[q^{CR},+\infty[$ with $q^{CR} \geq \hat{q}$. B is an interval too.

We begin the proof with the

PROPOSITION 3.5

Let $q_0 \in A$. Then $[q_0,+\infty[\subset A$.

If moreover $q_0 \in B$, $[q_0,+\infty[\subset B$.

REMARK 3.6

In the sequel, we will consider several times the equations (3.5) as an autonomous system :

$$\text{(3.8)} \qquad \begin{cases} u' = -su - sv + f(u) + C \\ v' = w \\ w' = -\frac{s}{d} w + \frac{K}{d} \phi(u)v. \end{cases}$$

If $[u(y),v(y),w(y)]$ is a maximal integral curve of (3.8) defined on an interval I, such that the estimates

$\forall\, y \in I$, $|u(y)| \le M$, $|v(y)| \le M$

hold, then the existence is global : $I = \mathbb{R}$. This is easily shown from the equation :

$$(w\, e^{\frac{sy}{d}})' = \frac{K}{d} \phi(u) v\, e^{\frac{sy}{d}} . \blacksquare$$

The proof of Proposition 3.5 consists in five lemmas. We are going to use the shooting argument mentioned in the introduction (see [2], [3] or [8]).

Let $q_0 \in A$ and $q_1 > q_0$. Let $\tilde{f} \in C^2(\mathbb{R})$ such that :

$$\begin{cases} \tilde{f} \text{ satisfies the assumptions (1.3) ,} \\ \forall\, u \ge U(q_1),\ \tilde{f}(u) = f(u) \ , \\ \tilde{f}(-\infty) > -\infty \ . \end{cases}$$

We denote $(\tilde{P})$ the problem (1.11) in which the function $\tilde{f}$ replaces f. Since $U(q_0) > U(q_1)$, the two problems (1.11) and $(\tilde{P})$ are equivalent when the energy value is q_0 or q_1. Therefore, in order to show that $q_1 \in A$, we are going to solve $(\tilde{P})$ for the energy q_1.

We first consider the case $q_0 > \hat{q}$. Let (u_0, v_0) be the solution of (1.11) satisfying : $v_0(+\infty) = q_0$, $u_0(0) = 0$.

For $m \in M = [\hat{q}, q_1 + 1]$, let (u_m, v_m) be defined by :

$$(3.9)\left\{\begin{array}{l} \left\{\begin{array}{l} u'_m = -su_m - sv_m + \tilde{f}(u_m) + C, \\ dv''_m + sv'_m = K\,\phi(u_m)v_m, \text{ on a maximal interval } D_m, \end{array}\right. \\ u_m(0) = 0 \;;\; v_m(0) = m \;;\; v'_m(0) = \dfrac{s}{d}(q_1 - m). \end{array}\right.$$

We have :

LEMMA 3.7

Let $m \in M$. Then :

$$\mathbb{R}_+ \subset D_m \;;\; v_m(+\infty) = q_1 \,.$$

Proof : If $m = \hat{q}$, we have : $u_m'(0) = 0$, $u_m''(0) < 0$; if $m > \hat{q}$, we get : $u_m'(0) < 0$. In both cases, $u_m(y)$ is negative for small positive y.

Assume : $\forall y \in D_m \cap \mathbb{R}_+^*$, $u_m(y) < 0$. Then :

$\forall y \in D_m \cap \mathbb{R}_+$, $v_m(y) = q_1 + (m - q_1)\, e^{\frac{-sy}{d}}$.

As v is monotone, we can apply the same argument as for Corollary 2.3 : If v is increasing on $D_m \cap \mathbb{R}_+$ $(m \le q_1)$, u is decreasing and : $\forall y \in D_m \cap \mathbb{R}_+$, $U(q_1) \le u(y) \le 0$. If v is decreasing $(m > q_1)$, we have :

$$\left\{\begin{array}{lll} \text{Either} & \forall y \in D_m \cap \mathbb{R}_+, & u'(y) < 0 \\ \text{Or} & \exists y_0 > 0, \forall y < y_0, & u'(y) < 0 \\ & \forall y > y_0, & u'(y) > 0,\ u(y) > u(y_0)\,. \end{array}\right.$$

If u is monotone, we still get $U(q_1) \le u \le 0$. Otherwise, it is easy to see that :

$$\forall\, y \in D_m \cap \mathbb{R}_+, \quad 0 \ge u(y) \ge u(y_0) \ge U(m).$$

It is then obvious to complete the proof : the assumption $u_m < 0$ holds, and $\mathbb{R}_+ \subset D_m$ from Remark 3.6 . ■

We define now :

$$(3.10) \quad \begin{cases} M_+ = \{ m \in M,\ \exists\, y \in D_m,\ v_m(y) > q_1 + 2 \} \\ M_- = \{ m \in M,\ \exists\, y \in D_m,\ v_m(y) < 0 \} \end{cases}$$

It follows from [2] that M_+ and M_- are open subsets of M.

Moreover, we have :

LEMMA 3.8

$$M_+ \cap M_- = \emptyset$$

Proof : Let $m \in M_+ \cap M_-$. We can write :

$$\begin{cases} \exists\, y_0 \in D_m \cap \mathbb{R}_-,\ v_m(y_0) > q_1 + 2,\ v'_m(y_0) < 0 ; \\ \exists\, y_2 \in D_m \cap \mathbb{R}_-,\ v_m(y_2) < 0 \quad\quad ,\ v'_m(y_2) > 0 . \end{cases}$$

If $y_0 < y_2$, we get :

$$\exists\, y_1 \in\,]y_0, y_2[,\ v_m(y_1) > q_1 + 2,\ v'_m(y_1) = 0,\ v''_m(y_1) \le 0,$$

whence $\phi(u_m(y_1)) \le 0$, $u_m(y_1) \le 0$. Using (3.8), it is then easily seen that : $\forall\ y > y_1$, $u_m(y) < 0$, $v_m(y) = v_m(y_1)$ which is impossible. We can argue in the same way if $y_0 > y_2$, and the proof is complete . ■

Let now $m_0 = v_0(0)$, $m_2 = q_1 + \frac{1}{2}$. Using Lemma 3.2, we get :

$$(3.11) \quad \forall\, y \in D_{m_0} \cap \mathbb{R}_-,\ v_0(y) \ge v_{m_0}(y) + (q_1 - q_0)\,(e^{-\frac{sy}{d}} - 1),$$

$$(3.12) \quad u_0(y) \ge u_{m_0}(y).$$

Then we have :

LEMMA 3.9

$m_0 \in M_-$,

Proof : If $\mathbb{R}_- \subset D_{m_0}$, (3.11) yields : $v_{m_0}(-\infty) = -\infty$, and $m_0 \in M_-$.
Assume now : $m_0 \notin M_-$. Then $D_{m_0} \cap \mathbb{R}_-$ is bounded and :

$$\forall\, y \in D_{m_0} \cap \mathbb{R}_-,\ 0 \leq v_{m_0}(y) \leq v_0(y) - (q_1 - q_0)\,(e^{-\frac{s\,y}{d}} - 1) :$$

v_{m_0} is bounded.

From (3.12), we have : $\forall y \in D_{m_0} \cap \mathbb{R}_-$, $u_{m_0}(y) \leq \max u_0 = M$
Since $\tilde{f}$ is bounded on the interval $]-\infty, M]$, we obtain :

$$\exists\, M' > 0,\ |\, u'_{m_0} + s\, u_{m_0} | = |\, \tilde{f}(u_{m_0}) - s\, v_{m_0} + C \,| \leq M'.$$

As $D_{m_0} \cap \mathbb{R}_-$ is bounded, this implies the boundedness of u_{m_0}. From Remark 3.6, we get : $\mathbb{R}_- \subset D_{m_0}$, whence a contradiction. ■

LEMMA 3.10

$m_2 \in M_+$

Proof : Arguing as in the proof of Lemmas 3.8 and 3.7, we obtain the monotonicity of v_{m_2} and u_{m_2} . If $m_2 \notin M_+$, it is obvious to see that v_{m_2} and u_{m_2} are bounded.
Then : $D_{m_2} = \mathbb{R}$; $+\infty > v_{m_2}(-\infty) > m_2 > 0$, $u_{m_2}(-\infty) > 0$, whence
$\lim\limits_{y \to -\infty} [d v''_{m_2}(y) + s\, v'_{m_2}(y)] > 0$, which is impossible. ■

We can generalize the arguments used above to show that

$[\hat{q}, m_0\,] \subset M_-$ $[m_2\,, q_1 + 1] \subset M_+$.

Since $\mathcal{M}_+$ and $\mathcal{M}_-$ are non empty disjoint open subsets of $\mathcal{M}$, we get :

$$\exists\ m_1 \in\ ]m_0, m_2[,\ m_1 \notin \mathcal{M}_+ \cup \mathcal{M}_- .$$

Setting $(u_1, v_1) = (u_{m_1}, v_{m_1})$, we have :

LEMMA 3.11

(u_1, v_1) is a solution of problem $(\tilde{P})$

Proof : a) Assume that the curve (u_1, v_1) hits the curve (u_0, v_0) :

$$\exists\ (y_0, y_1) \in \mathbb{R} \times \mathcal{D}_{m_1}, u_0(y_0) = u_1(y_1),\ v_0(y_0) = v_1(y_1) .$$

If $v'_0(y_0) = v'_1(y_1)$, we have :

$$\forall\ y \in \mathbb{R}, \quad v_1(y + y_1) = v_0(y + y_0),$$

and $q_1 = q_0$, which is impossible. If $v'_1(y_1) > v'_0(y_0)$, we can use Lemma3.2 and obtain, as above for m_0 : $m_1 \in \mathcal{M}_-$, which is again impossible. We leave it to the reader to check that $v'_1(y_1) < v'_0(y_0)$ implies : $m_1 \in \mathcal{M}_+$.

Therefore, the curves (u_0, v_0) and (u_1, v_1) do not hit each other. Since $m_1 > m_0$, we have : $v_1(0) > v_0(0)$: we will say that the curve (u_0, v_0) remains "under" the curve (u_1, v_1).

b) If : $\exists\ y \in \mathcal{D}_{m_1}$, $v'_1(y) < 0$, we can use the proof of Lemma 3.10 to get : $m_1 \in \mathcal{M}_+$. Therefore, we have,

$$\forall\ y \in \mathcal{D}_{m_1},\ v'_1(y) \geq 0, \quad 0 \leq v_1(y) \leq \quad q_1 + 2.$$

It is now easy to see that the alternative (3.1) holds for u_1. Then : u_1 is bounded, $\mathcal{D}_{m_1} = \mathbb{R}$. $u_1(-\infty)$ and $v_1(-\infty)$ exist, and : $u_1(-\infty) \geq u_0(-\infty) > 0$, from a) above. Since $\phi(u_1(-\infty)v_1(-\infty) = 0$, we get : $v_1(-\infty) = 0$, $u_1(-\infty) \in \{ A, B \}$ and finally $q_1 \in \mathcal{A}$. ■

To complete the proof of Proposition 3.5, it remains to consider the case $q_0 = \hat{q}$. Then it suffices to use the same arguments, with $m_0 = \hat{q}$. ■

Therefore we have proved that A and B are intervals.
We denote : $\inf A = q^{CR} \geq \hat{q}$. Before ending the proof of Theorem 3.4, we study some consequences of the shooting argument used in the preceding proof.

REMARK 3.12 : As indicated in the introduction, we can use the shooting argument to prove directly the existence result of Theorem 2.7. We now sketch this demonstration :

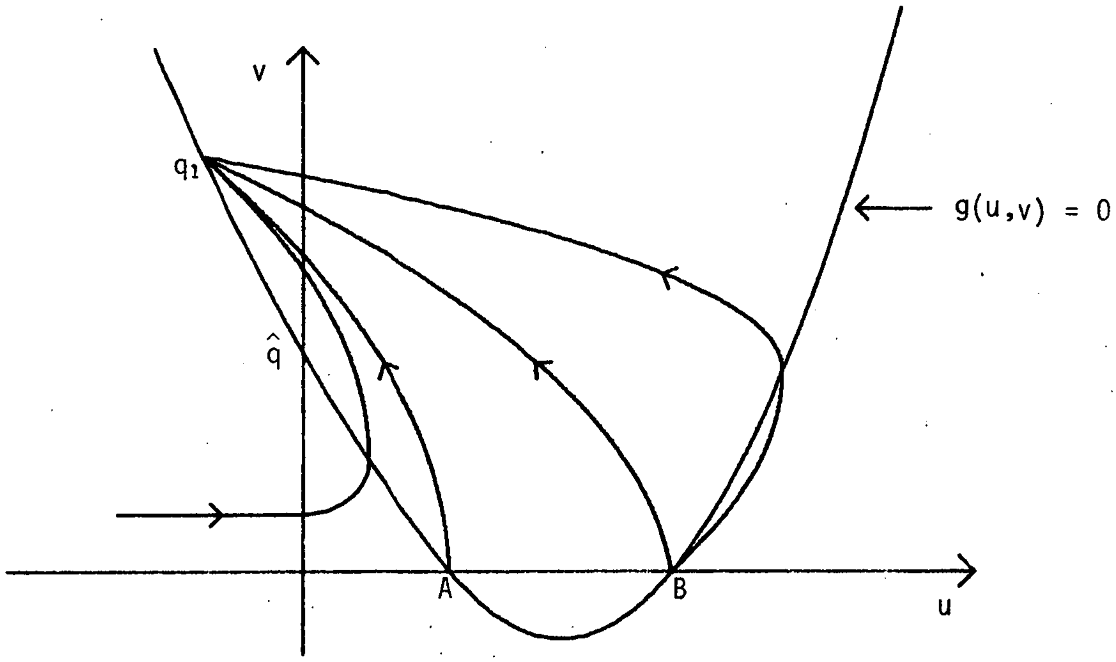
Consider $q_1 > \hat{q}$, and define (u_m, v_m), M_+ and M_- as in (3.9) and (3.10). Taking $m_0 = \hat{q}$, $m_2 = q_1 + 1$, we can show :

$\exists\ m_1 \in\ ]m_0, m_2[,\quad m_1 \notin M_+ \cup M_-$. Studying the properties of (u_{m_1}, v_{m_1}) and using Lemma 3.2 for the uniqueness, we get :

$$(3.13)\quad \begin{cases} \forall\ q_1 > \hat{q},\ \exists!\ (u_1,v_1) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \text{ such that :} \\ \qquad u_1' = -s\, u_1 - s\, v_1 + f(u_1) + C, \\ \quad dv_1'' + sv_1'' = K\, \Phi(u_1) v_1\ , \\ \quad v_1(+\infty) = q_1,\ u_1(+\infty) = U(q_1),\ u_1(0) = 0, \\ \forall\ y \in \mathbb{R},\ v'(y) \geq 0,\ v(y) \geq 0 \end{cases}$$

Fig. 2 shows the different possibilities for the solution (u_1,v_1) of (3.13).

Figure 2



It suffices now to use Lemmas 2.14 to 2.16, which are independent of the beginning of Section 2 : if q_1 satisfies :

$$q_1 > \left(1 + \frac{dr}{s}\right) \left[\hat{q} + \frac{rA}{s} + 2\left(\hat{q}\,\frac{rA}{s}\right)^{1/2}\right]$$

the solution (u_1,v_1) of (3.13) satisfies $u_1(-\infty) = B$, $v_1(-\infty) = 0$: (u_1,v_1) is a solution of (1.11). ■

We have actually proved a better result than Theorem 2.7, since we do not need to consider $\tilde{q}$! A consequence of Remark 3.12 is :

PROPOSITION 3.13 :

(i) The following inequality holds for q^{CR} :

$$q^{CR} \leq \left(1 + \frac{dr}{s}\right) \left[\hat{q} + \frac{rA}{s} + 2\left(\hat{q}\,\frac{rA}{s}\right)^{1/2}\right]. \tag{3.14}$$

(ii) Assume that q_0 satisfies :

$$q_0 > \left(1 + \frac{dr}{s}\right) \left[\hat{q} + \frac{rB}{s} + 2\left(\hat{q}\,\frac{rB}{s}\right)^{1/2}\right]. \tag{3.15}$$

Then, if (u,v) is the solution of (1.11), u is nonmonotone : u has a unique maximum, and this maximum exceeds B.

<u>Proof</u> : (3.14) is obvious. To prove (ii), it suffices to use Lemmas 2.14 to 2.16 with $a = B$: if (3.15) is assumed, the curve (u, v) does not hit the segment $\{0 \leq u \leq B \; ; \; v = \hat{q}\}$ and u is nonmonotone. ■

<u>REMARK 3.14</u> :

Another consequence of Remark 3.12 is the following : if the equation (1.6) has two solutions A, B with $A \leq 0 \leq B$, then for any positive energy q_0, (1.11) has a unique solution. (In this case, the solution of (3.13) is a solution of (1.11)). ■

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To complete the proof of Theorem 3.4, we need to prove the closedness of A. It will be a consequence of :

<u>THEOREM 3.15</u>

If $q_0 = q^{CR}$, there exists a unique weak detonation profile solution of (1.11). If $q_0 > q^{CR}$, the unique solution of (1.11) is a strong detonation profile.

The existence of a weak detonation profile was shown in a very simple way by A. Majda [7] in the case $d = 0$, using classical arguments for systems of two first-order ordinary differential equations. We are going to use the same method here ; the proof is rather lengthy since we are dealing with a system of three equations.

The proof is divided into five lemmas :

LEMMA 3.16 :

There exist exactly two integral curves (u,v,w) of (3.8) such that : $[u(-\infty), v(-\infty), w(-\infty)] = (A,0,0)$.

Proof : Computing the linearized system corresponding to (3.8) in the neighbourhood of the stationnary point (A,0,0), we obtain the matrice :

$$\begin{pmatrix} a(A) - s & -s & 0 \\ 0 & 0 & 1 \\ 0 & \frac{K}{d} & -\frac{s}{d} \end{pmatrix}$$

We have used (1.7) and (1.2) : $\phi(A) = 1$, $\phi'(A) = 0$. A short calculation gives the following eigenvalues :

(3.16) $\lambda_1 = r > 0$, $\lambda_2 = r' < 0$, $\lambda_3 = a(A) - s < 0$,

where r and r' satisfy $dr^2 + sr - K = 0$.

None of these eigenvalues has a real part equal to zero, and there is exactly one positive eigenvalue. The result is then an obvious consequence of the stable manifold theorem (see [4]) . ∎

LEMMA 3.17

There exist exactly two curves (u, v) such that :

(3.17) $\begin{cases} (u, v) \text{ satisfies the equations (3.5) on } \mathbb{R}_- \\ [u(-\infty), v(-\infty)] = (A, 0) \end{cases}$

Proof : Let (u, v) satisfy (3.17). (u, v) is the projection onto the (u, v) plane of a unique integral curve (u,v,v') of (3.8). It is then straightforward to show that $v'(-\infty) = 0$, and the proof is achieved. ∎

LEMMA 3.18

If $q_0 > q^{CR}$, the unique solution of (1.11) is a strong detonation profile.

Proof : It is obvious. If $q_0 \in A \setminus B$, we get $]q^{CR}, q_0[\subset A \setminus B$ from Proposition 3.5 and we have a continuum of solution for problem (3.17). ∎

LEMMA 3.19

Let (u, v) satisfy (3.17). Assume that, in the neighbourhood of (A,0) the curve (u, v) lies in the region

$$R = \{ u < A, \quad sv > -su + f(u) + C \}.$$

Then (u, v) is a solution of (1.11) for $q_0 = q^{CR}$.

Proof : Let (u,v) satisfy the above assumptions. Choosing one point of the curve (u, v) as $[u(0), v(0)]$, we denote by $\mathcal{D}$ the domain of values of y for which $[u(y), v(y)]$ is defined. Applying the same argument as for Lemma 3.8 we obtain the monotonicity of v. It is then easy to check that :

$$\forall y \in \mathcal{D}, \quad v'(y) \geq 0, \quad u'(y) < 0, \quad [u(y), v(y)] \in R$$

a) Assume $\exists\, y_1 \in \mathcal{D}$, $u(y_1) < 0$. The argument being exactly the same as for Lemma 3.7, we get $\mathcal{D} = \mathbb{R}$, and $v(+\infty) = q^{CR}$ from Lemma 3.18.

b) We will then suppose :

$$(3.18) \qquad \forall y \in \mathcal{D}, \ u(y) \geq 0$$

u is therefore bounded. In order to show that $\mathcal{D} = \mathbb{R}$, let us assume $\mathcal{D} =]-\infty, y_0[$, with $y_0 < +\infty$ and $\lim_{y \to y_0} v(y) = +\infty$

Then, from (3.18) : $\forall y \in \mathcal{D}$, $-s\,u(y) + f(u(y)) + C \leq f(0) + C = s\hat{q}$.

$$u'(y) \leq s\,\hat{q} - sv(y)$$

$$\phi\,(u(y)).u'(y) \leq s\hat{q}\,\phi\,(u(y)) - s\,\phi(u(y))v(y)$$

$$\phi\,(u(y))u'(y) \leq s\hat{q} - \frac{s^2}{K}\,v'(y) - \frac{s\,d}{K}\,v''(y)$$

After setting $F(u) = \int_0^u \Phi(u)du \geq 0$, we integrate the last inequality between 0 and $y > 0$:

$$F(u(y)) - F(u(0)) \leq s\hat{q}y - \frac{s^2}{K}[v(y) - v(0)] - \frac{sd}{K}[v'(y) - v'(0)] .$$

But $v'(y) \geq 0$, and we have : $\lim\limits_{y \to y_0} F(u(y)) = -\infty$ which is impossible.

We can therefore conclude that $D = \mathbb{R}$.

c) Still assuming (3.18), we get : $0 \leq u(+\infty) < A$, $v(+\infty) \in \mathbb{R}_+^* \cup \{+\infty\}$. If $v(+\infty) = +\infty$, we have $u'(+\infty) = -\infty$, which is impossible. Then : $0 < v(+\infty) < +\infty$, with $\Phi(u(+\infty))v(+\infty) = 0$ and $u'(+\infty) = 0$. Therefore : $u(+\infty) = 0$, $v(+\infty) = \hat{q} = q^{CR}$ and the proof is complete. ■

LEMMA 3.20

There exists exactly one curve (u, v) satisfying the assumptions of Lemma 3.19.

Proof : The uniqueness is a consequence of the stable manifold theorem. To prove the existence, we are going to use a linearization theorem for systems of ordinary differential equations, due to Samovol (see [1],[10]).

We can apply Samovol's result if :

$$(3.19) \qquad \lambda_1 + \lambda_2 \neq \lambda_3 \qquad \lambda_1 + \lambda_3 \neq \lambda_2$$

Using (3.16), (3.19) is equivalent to :

$$(3.20) \qquad d[s - a(A)] - s \neq 0, \ d^2[s - a(A)]^2 - s^2 - 4Kd \neq 0$$

a) If (3.20) is assumed, the two curves $[u(y), v(y)]$ approaching $(A,0)$ as $y \to -\infty$ are at this point tangential to the straight line directed by the eigenvector corresponding to the positive eigenvalue λ_1 and we can write :

$$(3.21) \qquad \lim_{y \to -\infty} \frac{u(y) - A}{v(y)} = \frac{s}{a(A) - s - r} < 0 .$$

In the neighbourhood of $(A,0)$, one of these curves lies in the region R.

b) Denote $\mathcal{S}$ the set of values of $(K,d) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ for which (3.20) does not hold. If $(K,d) \in \mathcal{S}$, we cannot use (3.21). The existence of a weak detonation profile is then a consequence of Theorem 4.1 below : $\mathcal{S}$ is of measure zero, and the existence is already proved for $(K,d) \notin \mathcal{S}$ ■

We have therefore proved Theorems 3.4 and 3.15. Of course if $q^{CR} > \hat{q}$, we have $\mathcal{B} =]\, q^{CR}, +\infty[$. But this is not necessarily true if $q^{CR} = \hat{q}$. We have no uniqueness result for the energy $\hat{q}$: there may exist a weak detonation profile connecting $(0,\hat{q})$ to $(A,0)$ <u>and</u> a strong detonation profile connecting $(0,\hat{q})$ to $(B,0)$.

SECTION IV : CONTINUOUS DEPENDANCE ASYMPTOTIC ANALYSIS

In this last section, we study the dependence of the solutions of (1.11) with respect to both parameters K and d. We will also show the relationship between (1.11) and A. Majda's problem (without diffusion) by studying the limit $d \to 0$.

The values of A,B,C and s are kept fixed, whereas K and d are now allowed to vary. We denote : $q^{CR} = q^{CR}(K,d)$. Our first result is concerning the weak detonation profiles :

THEOREM 4.1

a) Let $F \in]0, A[$. For $(K, d) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, define $[u(K,d,F), v(K,d,F)]$ as the unique weak detonation profile corresponding to (K, d), with $u(K,d,F)(0) = F$. Then the application :

$$\begin{cases} \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow C^2(\mathbb{R}) \times C^2(\mathbb{R}) \\ (K, d) \longrightarrow [u(K,d,F), v(K,d,F)] \end{cases}$$

is continuous.

b) Furthermore, the application

$$\begin{cases} \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+ \\ (K, d) \longrightarrow q^{CR}(K,d) \end{cases}$$

is continuous.

Proof : Let $(K, d) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, and consider two sequences of strictly positive reals (K_n) and (d_n) such that $\lim K_n = K$, $\lim d_n = d$.
We denote $u_n = u(K_n,d_n,F)$, $v_n = v(K_n,d_n,F)$.

a) Using (3.14), we can write :

(4.1) $\exists\, M_1 > 0,\ \forall\, n \in \mathbb{N},\ \forall\, y \in \mathbb{R},\ q \leq q^{CR}(K_n,d_n) \leq M_1$

(4.2) $\qquad 0 \leq v_n(y) \leq q^{CR}(K_n,d_n) \leq M_1$

(4.3) $\exists\, M_2 < 0, \qquad M_2 \leq U(q^{CR}(K_n,d_n)) \leq u_n(y) \leq A$

Since we obviously have : $v_n'(-\infty) = v_n'(+\infty) = 0$, we get :

$$\exists\, y_n \in \mathbb{R},\ v_n'(y_n) = \max_{y \in \mathbb{R}} v_n'(y),$$

and $v_n''(y_n) = 0$, $sv_n'(y_n) = K\Phi(u_n(y_n))v_n(y_n)$. Therefore :

(4.4) $\qquad \exists\, M_3 > 0,\ \forall\, n \in \mathbb{N},\ \forall\, y \in \mathbb{R},\ 0 < v_n'(y) \leq \frac{K_n}{s} M_1 \leq M_3$.

Finally, the equations (3.5) yield :

(4.5) $\qquad \exists\, M_4 < 0,\ \forall\, n \in \mathbb{N},\ \forall\, y \in \mathbb{R}, \qquad M_4 \leq u_n'(y) \leq 0$

(4.6) $\qquad \exists\, M_5 > 0, \qquad |u_n''(y)| < M_5$

(4.7) $\qquad \exists\, M_6 > 0, \qquad |v_n''(y)| < M_6$

Using the estimates (4.2) to (4.7), we deduce the existence of a subsequence (which we still denote (u_n,v_n)) such that :
$\lim\, (u_n,v_n,u_n',v_n',v_n'') = (u,v,u',v',v'')$ in $\mathcal{C}^0_{loc}(\mathbb{R})^5$.
(u,v) satisfies (3.5) and : $v' \geq 0, v \geq 0, u' \leq 0, u \leq A, u(0) = F$.
Furthermore, the curve (u,v) lies in the region R of Lemma 3.19. Then, we have : $v(-\infty) = 0$, $u(-\infty) = A$, whence : $(u,v) = [u(K,d,F), v(K,d,F)]$.
The limit of any subsequence is then uniquely determined : the whole sequence (u_n,v_n) is converging to (u,v) in $\mathcal{C}^1_{loc}(\mathbb{R}) \times \mathcal{C}^1_{loc}(\mathbb{R})$.

b) Using (4.1), we can extract a subsequence, still denoted (K_n,d_n), such that : $\lim q^{CR}(K_n,d_n) = Q \in [\hat{q}, M_1]$. We now want to show that $Q = q^{CR}(K, d)$.

c) Assume first : $q^{CR}(K,d) > \hat{q}$. Then : $\exists\, y_0 > 0$, $u(y_0) < 0$ and we have : $u_n(y_0) < 0$ for large n. Using (3.4) for v and v_n, we get :
$sv(y_0) + dv'(y_0) = sq^{CR}(K,d)$ and $sv_n(y_0) + dv_n'(y_0) = sq^{CR}(K_n,d_n)$. Taking the limit in the last equality, we obtain : $Q = q^{CR}(K, d)$.

d) We now assume : $Q > \hat{q}$. Then, for large values of n, $q^{CR}(K_n, d_n) > \hat{q}$. Extending slightly the definition of u(K,d,F), v(K,d,F), we can consider $\tilde{u}_n = u(K_n, d_n, 0)$, $\tilde{v}_n = v(K_n, d_n, 0)$.
Arguing as in a) above, we obtain the convergence of $(\tilde{u}_n, \tilde{v}_n)$ to $[u(K,d,0), v(K,d,0)]$. We can then replace y_0 by 0 in c) above to get the same result : $Q = q^{CR}(K, d)$.

e) This equality holds if $Q > \hat{q}$ and if $q^{CR}(K, d) > \hat{q}$. Since $Q \geq \hat{q}$ and $q^{CR}(K, d) \geq \hat{q}$, we have proved that $Q = q^{CR}(K, d)$.

f) Using the monotonicity of u_n, v_n, u, v, it is now obvious to prove the convergence of (u_n, v_n) to (u, v) in $\mathcal{C}^2(\mathbb{R}) \times \mathcal{C}^2(\mathbb{R})$. ■

An analogous result holds for strong detonation profiles :

<u>THEOREM 4.2</u>

Assume that the energy q_0 satisfies $q_0 > \hat{q}$.

a) Let $\mathcal{D}^* = \{(K,d) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, q^{CR}(K,d) < q_0\}$. $\mathcal{D}^*$ is a non empty open subset of $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

b) Let $F \in]\mathcal{U}(q_0), B[$. For $(K,d) \in \mathcal{D}^*$, define $[u(K,d,F), v(K,d,F)]$ as the unique strong detonation profile corresponding to (K,d), with $u(K,d,F)(0) = F$. Then the application

$$\begin{cases} \mathcal{D}^* \longrightarrow \mathcal{C}^2(\mathbb{R}) \times \mathcal{C}^2(\mathbb{R}) \\ (K, d) \longrightarrow [u(K,d,F), v(K,d,F)] \end{cases}$$

is continuous.

<u>REMARK 4.3</u>

We could denote $u(K,d,q_0,F)$, $v(K,d,q_0,F)$ with $(K,d,q_0,F) \in \{(K,d,q_0,F),\ q_0 > q^{CR}(K,d),\ \mathcal{U}(q_0) < F < B\} = \xi^*$. The preceding result is easily extended to this case. ■

Proof : a) $\mathcal{D}^*$ is obviously open. From (3.14), we have :

$$(4.8) \qquad \forall\, d > 0, \qquad \lim_{K \to 0} q^{CR}(K,d) = \hat{q},$$

which proves that $\mathcal{D}^*$ is non empty.

b) Let $(K,d) \in \mathcal{D}^*$. We use the notations K_n, d_n, u_n, v_n, as in the proof of Theorem 4.1. The convergence of a subsequence (u_n, v_n) in $\mathcal{C}^1_{loc}(\mathbb{R}) \times \mathcal{C}^1_{loc}(\mathbb{R})$ is straightforward. The limit (u,v) satisfies the equations (3.5). Then, we want to show that $v(+\infty) = q_0$, which together with $q_0 > q^{CR}(K,d)$, will imply $u(-\infty) = B$ and $(u,v) = [u(K,d,F), v(K,d,F)]$.

c) As in the proof of Theorem 4.1, the demonstration is easily finished if : $\exists\, y_0 \in \mathbb{R}$, $u(y_0) < 0$, or if $u(0) = F \le 0$: denoting $\tilde{u}_n = u(K_n, d_n, 0)$, $\tilde{v}_n = v(K_n, d_n, 0)$, we know that $(\tilde{u}_n, \tilde{v}_n)$ converges to $(\tilde{u}, \tilde{v}) = [u(K,d,0), v(K,d,0)]$ in $\mathcal{C}^1_{loc}(\mathbb{R}) \times \mathcal{C}^1_{loc}(\mathbb{R})$.

Assume $F > 0$. From Theorem 3.3 and Remark 1.1, we can write :

$\forall\, n \in \mathbb{N}$, $\exists\, y_n \in \mathbb{R}_-$, $u_n(y) = \tilde{u}_n(y + y_n)$, $v_n(y) = \tilde{v}_n(y + y_n)$.

y_n is determined by $\tilde{u}_n(y_n) = F$: it is then easy to show that $\lim y_n = y_1$, where y_1 satisfies $\tilde{u}(y_1) = F$. Hence, (u_n, v_n) converges in $\mathcal{C}^1_{loc}(\mathbb{R}) \times \mathcal{C}^1_{loc}(\mathbb{R})$ to $[\tilde{u}(\cdot + y_1), \tilde{v}(\cdot + y_1)]$, which is exactly $[u(K,d,F), v(K,d,F)]$.

d) It remains easy to show the convergence of (u_n, v_n) in $\mathcal{C}^2(\mathbb{R}) \times \mathcal{C}^2(\mathbb{R})$, although u and u_n are not necessarily monotone. ■

We now study the passage to the limit $d \to 0$.

THEOREM 4.4

Let $F \in \,]0,A[$, $K > 0$, $d \ge 0$, and define $[u(K,d,F), v(K,d,F)]$ as in Theorem 4.1. Then :

$$\left.\begin{array}{lll} u(K,d,F) & \longrightarrow & u(K,0,F) \text{ in } \mathcal{C}^1(\mathbb{R}) \\ v(K,d,F) & \longrightarrow & v(K,0,F) \text{ in } \mathcal{C}^0(\mathbb{R}) \end{array}\right\} \text{ as } d \to 0 .$$

Moreover, $\lim_{d \to 0} q^{CR}(K,d) = q^{CR}(K,0)$.

THEOREM 4.5

Let $K > 0$. Assume that the energy satisfies $q_0 > q^{CR}(K,0)$. Let $F \in]\,\mathcal{U}(q_0), B[$. For small values of $d > 0$, define $[u(K,d,F), v(K,d,F)]$ as in Theorem 4.2. Then :

$$\left.\begin{array}{lll} u(K,d,F) \longrightarrow u(K,0,F) & \text{in} & C^1(\mathbb{R}) \\ v(K,d,F) \longrightarrow v(K,0,F) & \text{in} & C^0(\mathbb{R}) \end{array}\right\} \quad \text{as } d \to 0.$$

Proof of Theorem 4.4 : Define K_n, d_n, u_n, v_n, as in the proof of Theorem 4.1 with $\lim d_n = 0$. The estimates (4.2) to (4.6) still hold, but we have no estimate for v_n''. We can extract a subsequence (u_n, v_n) converging to (u,v) in $C^1_{loc}(\mathbb{R}) \times C^0_{loc}(\mathbb{R})$.

Then : $u' = -su - sv + f(u) + C$.

Let $\psi \in \mathcal{D}(\mathbb{R})$, and $I = \operatorname{supp} \psi$. Thus :

$$\int_I (d_n v_n'' \psi + s v_n' \psi - K_n \Phi(u_n) v_n \psi)\,dy = 0$$

or

$$\int_I (d_n v_n \psi'' - s v_n \psi' - K_n \Phi(u_n) v_n \psi)\,dy = 0$$

Since $(u_n, v_n) \to (u,v)$ in $C^0(I)^2$ and $d_n \to 0$, we get :

$$\int_I (-s v \psi' - K \Phi(u) v \psi)\,dy = 0$$

whence $sv' = K \Phi(u) v$ in the sense of differentiation in $\mathcal{D}'(\mathbb{R})$. Therefore v' is a continuous function : it is the derivative of v in the classical sense. Then : $v \in C^1(\mathbb{R})$ and $sv' = K \Phi(u) v$.

The end of the proofs of Theorems 4.4 and 4.5 is now obvious and will be omitted. ■

We end the paper with some other properties of the function $q^{CR}(K,d)$:

PROPOSITION 4.6

Let $d > 0$. $q^{CR}(.,d)$ is a monotone increasing function on $\mathbb{R}_+^*$. If $q^{CR}(K_0,d) > \hat{q}$, the function is strictly increasing on $[K_0,+\infty[$. Furthermore, we have :

$$\lim_{K \to 0} q^{CR}(K,d) = \hat{q}, \quad \lim_{K \to +\infty} q^{CR}(K,d) = +\infty$$

We begin the proof with a new comparison lemma, which is the analogue of Lemma 3.2 for different values of K :

LEMMA 4.7 :

Let $d > 0$ and $K_1 > K_0 > 0$. For $i \in \{0,1\}$, let (u_i,v_i) satisfy :

$$\begin{cases} u_i' = - su_i - sv_i + f(u_i) + C , \\ dv_i'' + sv_i' = K_i \ \Phi \ (u_i)v_i , \end{cases}$$

with : $0 \leq u_0(0) \leq u_1(0)$; $0 < v_0(0) \leq v_1(0)$; $0 \leq v_1'(0) \leq v_0'(0)$.

If the equality :

(4.9) $\qquad [u_0(0), v_0(0), v_0'(0)] = [u_1(0), v_1(0), v_1'(0)]$

holds, assume that : $u_0(0) > 0$ or $u_0'(0) < 0$. Then :

$$\forall\ y < 0,\ u_0(y) < u_1(y), \quad v_0(y) < v_1(y).$$

Moreover : $\displaystyle\lim_{y \to -\infty} [v_1(y) - v_0(y)] = +\infty$.

Proof : It is very similar to the demonstration of Lemma 3.2 and is left to the reader. The only new difficulty is to show that the result remains true when (4.9) holds. ■

LEMMA 4.8 :

Let $d > 0$, and $K_1 > K_0 > 0$. Then

$$q^{CR}(K_1,d) \geq q^{CR}(K_0,d).$$

Proof : a) Define $A_i = [q^{CR}(K_i,d), +\infty[$. Let $q_1 \in A_1 \cap]\hat{q}, +\infty[$. We denote (u_1,v_1) the corresponding solution with $u_1(0) = 0$. It suffices to show that $q_1 \in A_0$. We define u_m, v_m, M_+, M_-, as in (3.9) and (3.10). For $m_1 = v_1(0)$, we can use Lemma 4.7, and conclude that $m_1 \in M_-$. Since $q_1 + \frac{1}{2} \subset M_+$, we obtain the existence of $m_0 \notin M_+ \cup M_-$. We set $(u_{m_0}, v_{m_0}) = (u_0, v_0)$. $m_0 > m_1$ yields $v_0(0) > v_1(0)$.

b) Assume now that the curve (u_0,v_0) hits the curve (u_1,v_1) in the region $\{u \geq 0\}$. We consider :

$y_0 = \max \{ y \in D_{m_0} \cap \mathbb{R}_-, \exists z \in \mathbb{R}, u_0(y) = u_1(z), v_0(y) = v_1(z) \}$.

y_0 corresponds to the first intersection point, when the curves are followed in the direction of decreasing v. We set : $u_0(y_0) = u_1(y_1)$, $v_0(y_0) = v_1(y_1)$, and assume : $u_0'(y_0) = u_1'(y_1) < 0$.

As $v_0(0) > v_1(0)$, we have

$$\frac{v_0'(y_0)}{|u_0'(y_0)|} \geq \frac{v_1'(y_1)}{|u_1'(y_1)|}$$

or $v_0'(y_0) \geq v_1'(y_1) \geq 0$. We can then use again Lemma 4.7 and get $m_0 \in M_-$ whence a contradiction. Therefore, using (3.1), we can state : the curve (u_0,v_0) cannot hit the curve (u_1,v_1) at a point where $u_0 \geq 0$, $u_0' < 0$.

c) We can then end the proof as we did for Proposition 3.5 : (u_0,v_0) is a solution of (1.11) corresponding to q_1 and K_0 . Thus $q_1 \in A_0$ and the proof is complete. (The details are left to the reader). ■

REMARK 4.9

It will be useful in the sequel to notice the following : assume $K_1 > K_0$ and $q^{CR}(K_1,d) = q^{CR}(K_0,d) > \hat{q}$. Then, in the region $\{u \geq 0\}$, the weak detonation profile corresponding to K_1 is strictly under the weak detonation profile corresponding to K_0 . This is a consequence of the preceding proof, since $u' < 0$ along weak detonation profiles. ■

We can now complete the proof of Proposition 4.6 with the two next lemmas :

LEMMA 4.10

Let $d > 0$ and $K_0 > 0$, such that $q^{CR}(K_0,d) > \hat{q}$. Then the function $q^{CR}(.,d)$ is strictly increasing on the interval $[K_0, +\infty[$.

Proof : In fact, we are going to prove the next statement :

" For $K_2 > K_1 > 0$, the weak detonation curve (u_1,v_1) corresponding to K_1 is under the weak detonation curve (u_2,v_2) corresponding to K_2 in the region $\{u \geq 0\}$ ", and Lemma 4.10 will be proved from Remark 4.9.

a) Assume that (3.20) holds for (K_1,d) and (K_2,d). We can then use (3.21): in the neighbourhood of $(A,0)$, the curve (u_1,v_1) is under the curve (u_2,v_2). Again from Lemma 4.7, the curve (u_1,v_1) remains under the curve (u_2,v_2) in the region $\{u \geq 0\}$.

b) If $(K_1,d) \in S$ or $(K_2,d) \in S$ ((3.20) does not hold), we take two sequences (K_n^1, d_n), (K_n^2, d_n) such that :

$$\begin{cases} \lim K_n^1 = K_1 \quad , \quad \lim K_n^2 = K_2 \quad , \lim d_n = d \; ; \\ \forall\, n \in \mathbb{N} \, , \quad (K_n^1, d_n) \notin S \; , \quad (K_n^2, d_n) \notin S \; . \end{cases}$$

It suffices now to use Theorem 4.1 and a) above , and the proof is complete. ■

LEMMA 4.11

Let $d > 0$. Then

$$\lim_{K \to 0} q^{CR}(K,d) = \hat{q} \qquad \lim_{K \to +\infty} q^{CR}(K,d) = +\infty$$

Proof : We already proved the first limit (see (4.8)). From Lemma 4.8,

$$\lim_{K \to +\infty} q^{CR}(K,d) \text{ exists in } \overline{\mathbb{R}}_+ \; .$$

Assume : $\lim_{K \to +\infty} q^{CR}(K,d) = M < +\infty$

Let $F \in]U_0, A[$ $(U_0 = \inf\{u, \ \Phi(u) = 1\})$, let $K_2 > K_1 > 0$. Using the notations of Theorem 4.1, we set : $u_i = u(K_i, d, F)$, $v_i = v(K_i, d, F)$ for $i \in \{1, 2\}$. Since the curve (u_1, v_1) is under the curve (u_2, v_2) in the region $\{u \geq 0\}$, we have : $u_1(0) = u_0(0) = F$ and $v_1(0) < v_2(0)$, whence $u_1'(0) > u_2'(0)$.

Then : $u_1(y) > u_2(y)$ and $v_1(y) < v_2(y)$ for small positive y. We define $y_2 > 0$ by : $u_2(y_2) = U_0 < F$. Using (3.3), one can easily prove that, for $y \in [0, y_2]$:

$$A > u_1(y) > u_2(y) \geq U_0 ,$$

$$v_1(y) < v_2(y) \leq M$$

Thus : $u_2'(y) \geq -sM - su_2(y) + f(u_2(y)) + C \geq -sM$.

We get by integration :

$$u_2(y_2) - u_2(0) = U_0 - F \geq -sM\, y_2,$$

whence $$y_2 \geq \frac{F - U_0}{sM} = y_1 .$$

We define r_i by $dr_i^2 + sr_i - K_i = 0$, $r_i > 0$ for $i \in \{1,2\}$.

We have from (3.3) :

$$\forall\, y \in [0, y_1] , \ v_2'(y) = r_2 v_2(y) > r_2 v_1(y) > \frac{r_2}{r_1} v_1'(y).$$

Therefore : $$v_2(y_1) \geq v_2(0) + \frac{r_2}{r_1}[(v_1(y_1) - v_1(0)] ,$$

$$q^{CR}(K_2,d) \geq v_2(y_1) \geq \frac{r_2}{r_1}[v_1(y_1) - v_1(0)] .$$

The difference $v_1(y_1) - v_1(0) > 0$ does not depend on K_2 .

Since $r_2 = \dfrac{-s + \sqrt{s^2 + 4dK_2}}{d}$, we get : $\lim_{K_2 \to +\infty} q^{CR}(K_2,d) = +\infty$

and the proof is achieved. ■

APPENDIX

We now consider the case d = 0, which has been investigated by A. Majda [7] . We study the continuous dependence of the combustion profiles with respect to K and we propose another formulation for A. Majda's results.

1) Consider the equations :
$$\begin{cases} u' = - su - sv + f(u) + C = f_1(u, v) \\ v' = \dfrac{K}{s} \cdot \Phi(u)v = f_2(u,v\ ;\ K). \end{cases}$$

Let $K_1 > K_0 > 0$. On Fig. 3, the curve AD represents the weak detonation profile corresponding to $\underline{K_0}$ and the arrows represent the direction of the vectors $[\ f_1(u,v)\ ,\ f_2(u,v\ ;\ \underline{K_1}\ )\ ]$.

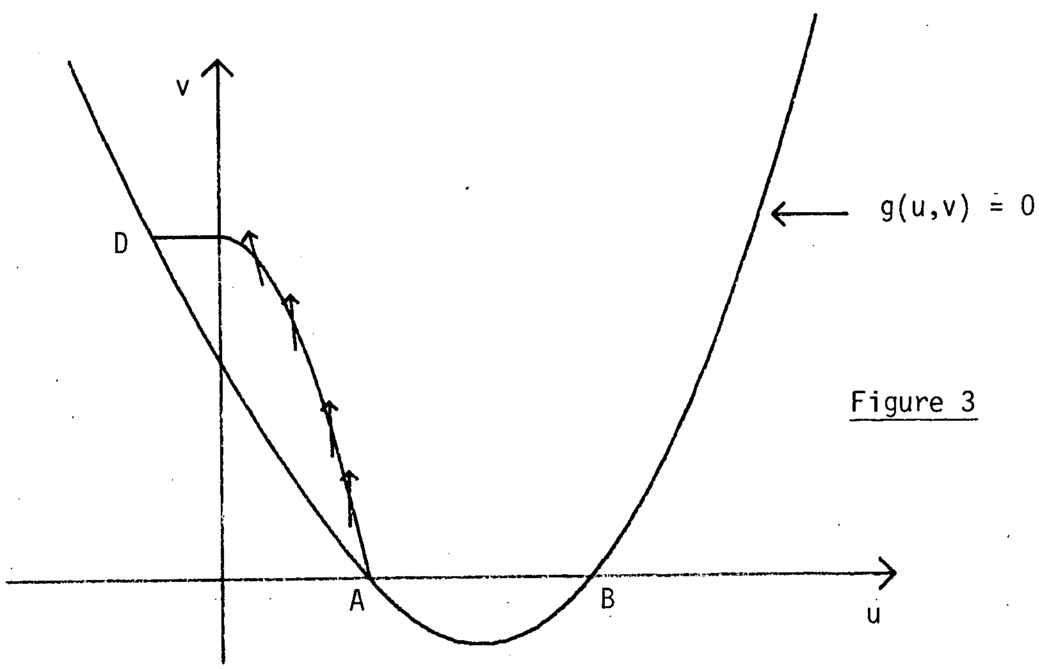

Figure 3

PROPOSITION A.1

The function $q^{CR}(.,0)$ is increasing. If $q^{CR}(K_0,0) > \hat{q}$, the function is strictly increasing on the interval $[K_0, +\infty[$.

Proof : In the neighbourhood of (A,0), the weak detonation curve corresponding to K_1 is above the weak detonation curve corresponding to K_0 (see [7]). The proposition is then obvious, from Fig. 3. ■

2) The proof of the next result is easy and will be omitted.

PROPOSITION A.2

a) With the notations of Theorem 4.1, the application

$$\begin{cases} \mathbb{R}_+^* \times \underline{\mathbb{R}_+} \longrightarrow \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^0(\mathbb{R}) \times \mathbb{R} \\ (K, d) \longrightarrow [u(K,d,F), v(K,d,F), q^{CR}(K,d)] \end{cases}$$ is continuous .

b) Assume that the energy satisfies $q_0 > \hat{q}$, and define $\mathcal{D} = \{(K, d) \in \mathbb{R}_+^* \times \underline{\mathbb{R}_+}, q^{CR}(K,d) < q_0\}$. $\mathcal{D}$ is a non empty open subset of $\mathbb{R}_+^* \times \mathbb{R}_+$; with the notations of Theorem 4.2, the application

$$\begin{cases} \mathcal{D} \longrightarrow \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^0(\mathbb{R}) \\ (K,d) \longrightarrow [u(K,d,F), v(K,d,F)] \end{cases}$$ is continuous. ■

3) A consequence of Propositions A.1 and A.2 is :

$$\text{(A.1)} \quad \begin{cases} \forall q_0 > \hat{q}, \exists! K_0 > 0, q^{CR}(K_0, 0) = q_0 ; \\ \forall K < K_0, q^{CR}(K,0) < q_0 ; \quad \forall K > K_0, q^{CR}(K,0) > q_0 . \end{cases}$$

This is true even if $A \in]0,U_0]$: the assumption $A > U_0$ is not useful to prove $\lim\limits_{K \to +\infty} q^{CR}(K,0) = +\infty$ (see [7]).

We can then restate A. Majda's results in a somewhat more general manner than in Theorem 1.2, where A, B, C and s are fixed, while the energy q_0 and therefore the state $U\,(q_0)$ of the cold mixture are allowed to vary.

Let be given the value q_0 of the liberated energy and the state $U \leq 0$ of the cold mixture. We consider the problem :

$$\text{(A.2)}\quad \begin{cases} \text{For } s > 0, \text{ find } (u,v) \in C^1(\mathbb{R}) \times C^1(\mathbb{R}) \text{ and } U' > 0 \text{ such that :} \\ u' = -su - sv + f(u) + [sU + sq_0 - f(U)] \;, \\ v' = \dfrac{K}{s}\,\Phi(u)v \;, \\ v(-\infty) = 0 \;,\; v(+\infty) = q_0 \;, \\ u(-\infty) = U' \;,\; u(+\infty) = U \;. \end{cases}$$

If (u, v, U') is a solution of (A.2), we have $s = \dfrac{f(U') - f(U)}{U' - (U + q_0)}$.

We sill state the result for the case represented on Fig. 4 : $U + q_0 > 0$. The other cases can be treated without difficulty .

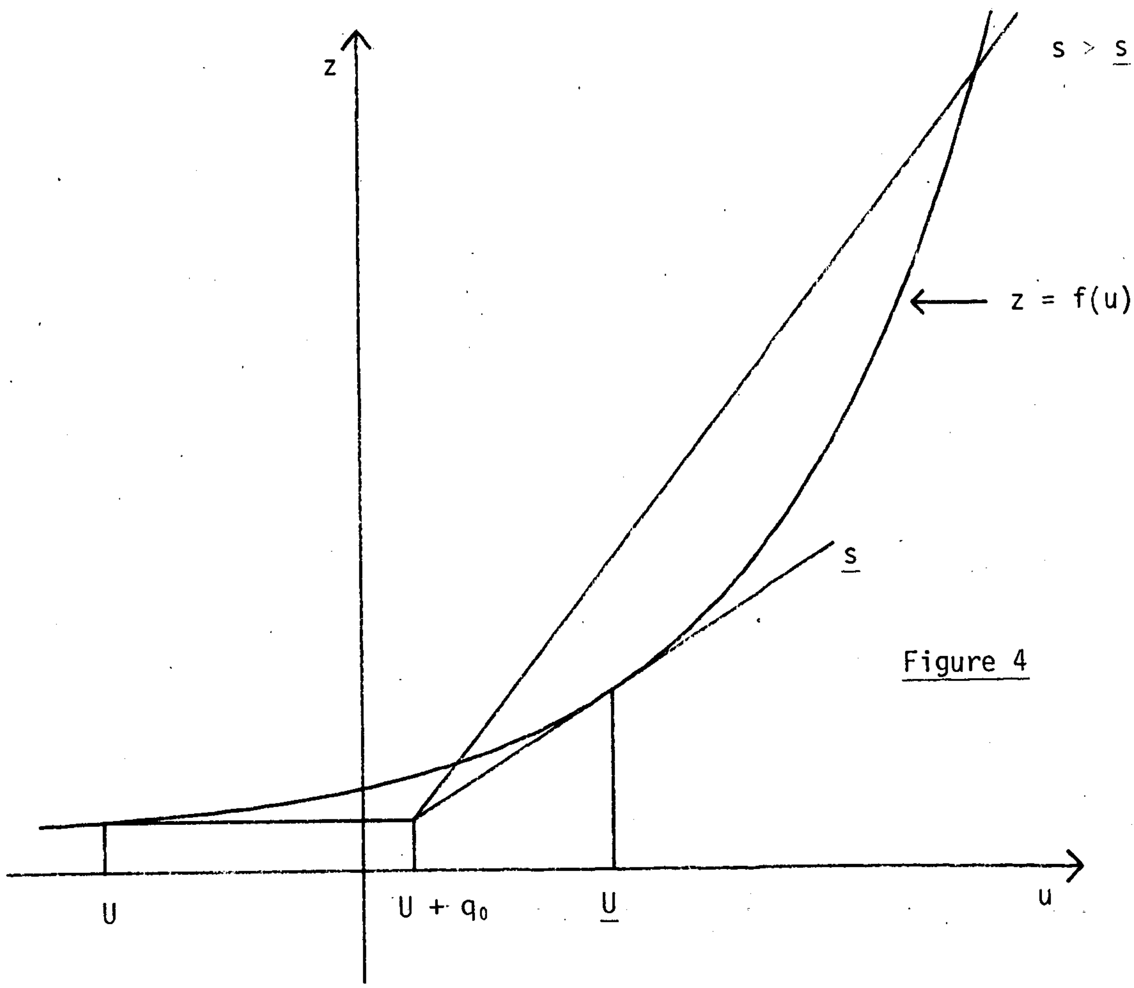

Figure 4

PROPOSTION A.3

There exists a critical value $\underline{s}$ for the wave speed s such that :

a) If $s < \underline{s}$, (A.2) has no solution.

b) If $s = \underline{s}$, there exists a unique solution for all positive K , with $U' = \underline{U}$.

c) If $s > \underline{s}$, there exists $K_0(s) > 0$ such that :

(i) There exists a unique solution with $U' > \underline{U}$ if $K < K_0(s)$.

(ii) There exists a unique solution with $U' < \underline{U}$ if $K = K_0(s)$.

(iii) No solution exists if $K > K_0(s)$. ■

The proof is an easy consequence of Theorem 1.2, Remark 1.6 , together with (A.1) and Fig. 4 . Notice that the solutions of (i) above are strong detonations, whereas the solution of (ii) is a weak detonation.

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Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique